# Dale Mesner and his contributions to Algebraic Combinatorics

Robert Jajcay
Comenius University and University of Primorska

Symmetry vs. regularity
Pilsen
July 5, 2018

# Dale Marsh Mesner

1923 - 2009

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- May 22, 1956, defended dissertation under the advisement of Leo Katz, entitled: *An Investigation of Certain Combinatorial Properties of Partially Balanced Incomplete Block Experimental Designs and Association Schemes, with a Detailed Study of Designs of Latin Squares and Related Types*

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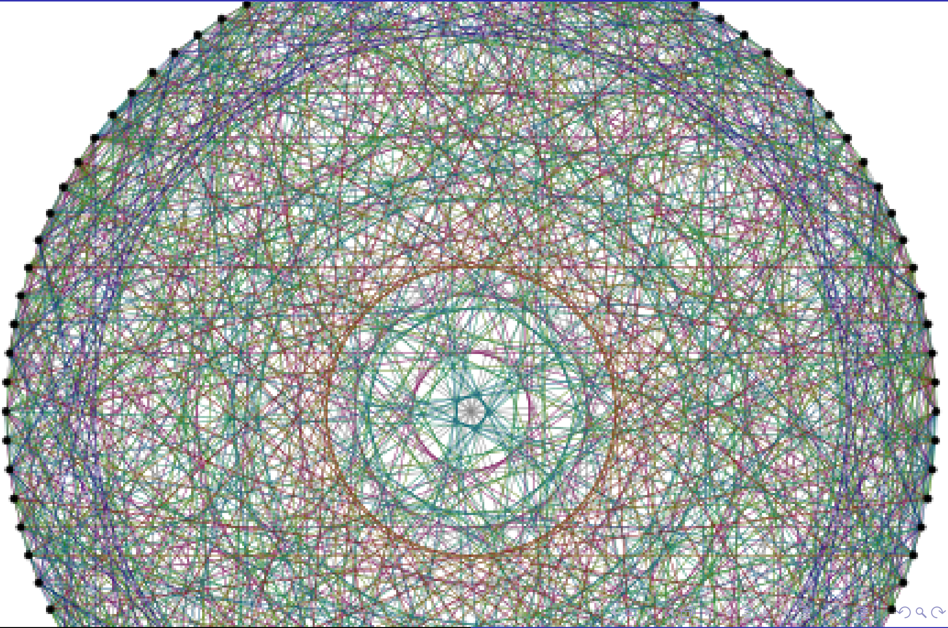
- he also did not consider any automorphism groups

# Higman-Sims Graph



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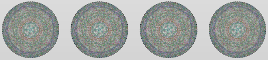

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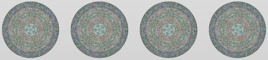

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E. Bannai, R.L. Griess Jr., C.E. Praeger, and L. Scott, **The Mathematics of Donald Gordon Higman**, *Michigan Math. J.* 58 (2009).

M.H. Klin and A.J. Woldar, **Dale Mesner, Higman & Sims, and the strongly regular graph with parameters (100,22,0,6)**, *Bull. Inst. Comb. Appl.* 63, 13-35 (2011).

# Bose-Mesner Algebras

## Definition

- A $k$-class **association scheme** $\mathcal{A}$ on a set $X$ consists of $k+1$ non-empty symmetric binary relations $R_0 = \{(x,x) \mid x \in X\}, R_1, \ldots, R_k$ on $X$ which partition the product $X \times X$ and satisfy the "regularity" condition: for each $0 \leq \ell, i, j \leq k$, there exists a nonnegative integer $p_{ij}^{\ell}$ such that for any $(x,y) \in R_\ell$ there are exactly $p_{ij}^{\ell}$ elements $z \in X$ such that $(x,z) \in R_i$ and $(z,y) \in R_j$

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- the linear span of the matrices $A_i$, $0 \leq i \leq k$, over the reals, is the **Bose-Mesner algebra** of the scheme

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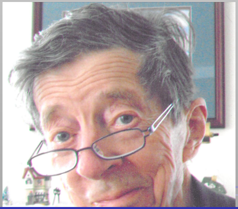
Introducing the Bose-Mesner algebras of these schemes allows one the use of the powerful tools of linear algebra, and naturally yields many beautiful results.

Today, the concept of a Bose-Mesner algebra has become a "text-book item" of relevance to anyone interested in combinatorics.



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- 1964-66, stayed at UNC, Department of Mathematics, two papers with M. E. Watkins

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# Kramer-Mesner matrices

Kramer-Mesner matrices are incidence matrices that allow one to construct $t$-designs invariant under a prescribed group.

- a $t-(v,k,\lambda)$ design $(X,\mathcal{B})$ is **$G$-invariant** if the image of every $k$-block $B \in \mathcal{B}$ is again a block in $\mathcal{B}$
- the Kramer-Mesner matrices formalize the precise conditions under which $G$-orbits on $X_k$ can be selected to form a $t-(v,k,\lambda)$ design

**Theorem**

*Let* $\mathbf{A} = A_{t,k}$ *denote the* $\rho_t \times \rho_k$ *matrix whose entries* $a_{ij}$ *count the number of* $k$*-subsets in the* $j$*-th orbit of the action of* $G$ *on* $X_k$ *that contain (any) fixed* $t$*-subset from the* $i$*-th orbit of* $G$ *on* $X_t$*.*

*There exists a* $G$*-invariant* $t-(v,k,\lambda)$ *design* $(X,\mathcal{B})$ *if and only if there exists a solution vector* $\mathbf{u}$ *to the matrix equation*

$$\mathbf{Au} = \lambda \mathbf{J},$$

*where* $\mathbf{A} = A_{t,k}$*,* $\mathbf{u}$ *is* $\rho_k$*-dimensional vector of non-negative integral entries,* $\mathbf{J}$ *is the vector of all 1's, and* $\lambda$ *is a positive integer.*

*The* $t$*-design is simple iff the vector* $\mathbf{u}$ *is 0-1.*

# Embedding Arbitrary Graphs into Strongly Regular Graphs

Given a finite simple graph $\Gamma$, find a 'small' strongly regular $\tilde{\Gamma}$ such that $\Gamma$ is isomorphic to an induced subgraph of $\tilde{\Gamma}$.

# SRG's based on Desarguesian affine plane geometries

## Definition

1. Let $F$ be a finite field, and consider the Desarguesian affine plane geometry coordinatized by $F$. It contains $n^2$ points $(x, y)$, $n^2$ lines of the form $y = mx + b$, $m, b \in F$, and $n$ vertical lines; all the lines contain $n$ points.

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2. The vertices of the **strongly regular graph based on the Desarguesian affine plane** $F^2$ are the $n^2$ points $(x, y)$. To define the edges, choose any $2 \leq g \leq n + 1$, and any $g$ of the $n + 1$ parallel classes of lines in the geometry. Two vertices of the graph are adjacent iff the line joining them in the geometry is one of the selected lines.

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$$srg(n^2, g(n-1), n-2+(g-1)(g-2), g(g-1)).$$

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5. the bijection $\phi : V(\Gamma) \to S$ is an embedding of $\Gamma$ into $\tilde{\Gamma}$.

# Sidon Sets

**Definition (Sidon, 1933)**

Let $G$ be a finite abelian group. A subset $D = \{x_1, x_2, \ldots, x_k\}$ of $G$ is a **Sidon set** in $G$ provided the set of sums $\{x_i + x_j | x_i, x_j \in D, i \leq j\}$ of pairs of elements from $D$ consists of $(k(k-1)/2) + k$ distinct elements of $G$ (i.e. no two of the sums are equal).

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**Lemma (Erdös, Turán, 1941)**

*Let $G$ be a finite cyclic group of order $n$. Then $G$ contains a Sidon set $D$ of size $cn^{1/2}$.*

# Embedding Finite Simple Graphs into Small SRG's

Theorem (RJ, Mesner)

*If $\Gamma$ is a finite simple graph on $v$ vertices, then there exists a strongly regular graph $\tilde{\Gamma}$ on $O(v^4)$ vertices that contains an induced subgraph isomorphic to $\Gamma$.*

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4. all the lines determined by pairs of points in $S$ have different slopes:

$$(x_{i_2}^2 - x_{i_1}^2)/(x_{i_2} - x_{i_1}) = x_{i_2} + x_{i_1}$$
$$(x_{i_4}^2 - x_{i_3}^2)/(x_{i_4} - x_{i_3}) = x_{i_4} + x_{i_3},$$


# Thank You


26. 8. – 30. 8. 2019
EUROCOMB 2019 in Bratislava
Algebraic graph theory is welcome!

Barcelona, Berlin, Bordeaux, Budapest, Bergen

# Bratislava!