Geometric structures on the complement of a knotted θ-graph embedded in $\mathbb{S}^3$

Nikolay Abrosimov

(Sobolev Institute of Mathematics / Novosibirsk State University)

joint work with Alexander Mednykh and Dasha Sokolova

Conference in Algebraic Graph Theory
Symmetry vs Regularity
Pilsen, Czech Republic
July 4, 2018

Examples of geom. structures on knots complements in $\mathbb{S}^3$

1975 R. Riley found first examples of hyperbolic structures on seven *excellent* knots and links in $\mathbb{S}^3$.

1977 W. Thurston showed that a complement of any prime knot admits a hyperbolic structure if this knot is not toric or satellite one.

1980 W. Thurston constructed a hyperbolic 3-manifold homeomorphic to the complement of knot 4_1 in $\mathbb{S}^3$ by gluing faces of two regular ideal tetrahedra. This manifold has a complete hyperbolic structure.

1982 J. Minkus suggested a general topological construction for the orbifold whose singular set is a two-bridge knot in $\mathbb{S}^3$.

1998/2006 A. Mednykh, A. Rasskazov found a geometrical realisation of the Minkus construction in $\mathbb{H}^3, \mathbb{S}^3, \mathbb{E}^3$.

2009 E. Molnár, J. Szirmai, A. Vesnin realised the figure-eight knot cone-manifold in the five exotic Thurston's geometries.

2004 H. Hilden, J. Montesinos, D. Tejada, M. Toro considered more general construction known as *butterfly*.

Upper half-space model of hyperbolic 3-space

Denote by $\mathbb{H}^3$ a 3-dim *hyperbolic space* (*Lobachevsky–Bolyai–Gauss space*). $\mathbb{H}^3$ can be modelled in $\mathbb{R}^3_+ = \{(x, y, t) \,:\, x, y, t \in \mathbb{R}, t > 0\}$ with metric s given by expression $ds^2 = \dfrac{dx^2 + dy^2 + dt^2}{t^2}$.
The boundary $\partial\mathbb{H}^3 = \{(x, y, 0) \,:\, x, y \in \mathbb{R}\}$ caled *absolute* and consist of points at infinity.

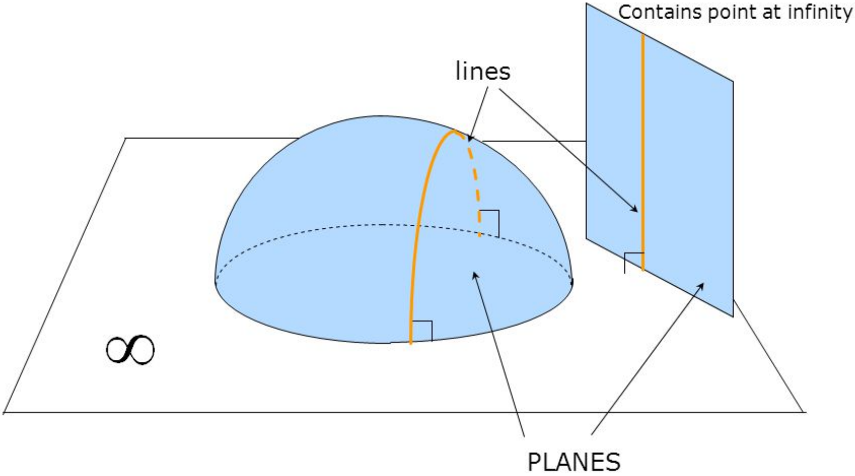
Isometry group $\mathrm{Isom}(\mathbb{H}^3)$ is a group of all actions on $\mathbb{H}^3$ preserving the metric s. Denote by $\mathrm{Isom}^+(\mathbb{H}^3)$ the group of orientation preserving isometries.

$\mathrm{Isom}^+(\mathbb{H}^3) \cong \mathbf{PSL}(2, \mathbb{C})$ (Pozitive Special Lorentz group). An element $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbf{PSL}(2, \mathbb{C})$ acts on $\mathbb{H}^3$ by the rule

$$g \,:\, (z, t) \mapsto \left(\frac{(az + b)\overline{(cz + d)} + a\bar{c}t^2}{|cz + d|^2 + |c|^2\, t^2}, \frac{t}{|cz + d|^2 + |c|^2\, t^2} \right),$$

where $z = x + i\, y$.

Geodesic lines and planes in half-space model of $\mathbb{H}^3$



$\mathrm{Isom}(\mathbb{H}^3)$ is generated by reflections with respect to geodesic planes.

Manifolds & orbifolds

Definition

Manifolds and *orbifolds* having a complete geometric structure can be presented as the quotient space X/Γ, where X is one of known geometries and Γ is a discrete isometry group acting on X with fixed points in general.

2-dim case: $X = \mathbb{S}^2, \mathbb{E}^2$ or $\mathbb{H}^2$.

3-dim case: $X = \mathbb{S}^3, \mathbb{E}^3, \mathbb{H}^3, \mathbb{S}^2 \oplus \mathbb{E}^1, \mathbb{H}^2 \oplus \mathbb{E}^1, Nil, Sol, \widetilde{PSL(2, \mathbb{R})}$.

In 3-dim case the image of fixed points of group Γ under canonical map $X \to X/\Gamma$ is generally a knot, link or knotted graph. In n-dim case it has dimension $(n-2)$. Today we consider only 3-dim case.

Example (Hilden, Lozano, Motesinos, 92)

Let $X = \mathbb{H}^3$ and $\Gamma = F_{2n} = \langle a_1, \ldots, a_{2n} : a_j a_{j+1} = a_{j+2}, j \mod 2n \rangle, n \geq 4$ is Fibonacci group acting on X by isometries. Then X/Γ is 3-dimensional sphere and the image of fixed points of X in X/Γ is the figure-eight knot.

Cone-manifolds

In general, the presence of a geometrical structure is not necessarily associated with discrete groups. As a result a cone-manifold arises, which can be viewed as a direct generalization of orbifold. In turn, in the definition of cone-manifold, we require just a local uniformization with the above geometries.

Definition

A *Euclidean cone-manifold* is a metric space obtained as the quotient space of a disjoint union of a collection of geodesic n-simplices in $\mathbb{E}^n$ by an isometric pairing of codimension-one faces in such a combinatorial fashion that the underlying topological space is a manifold. *Hyperbolic* and *spherical cone-manifolds* are defined similarly.

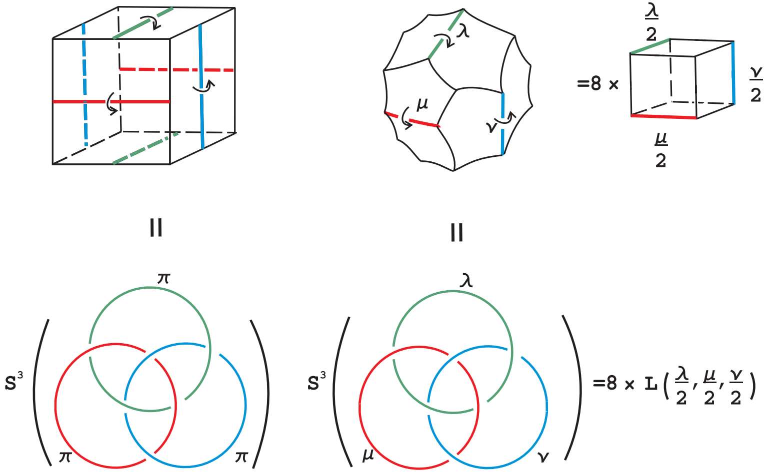
The metric structure near each 1-cell is determined by the *conical angle*, which is the sum of dihedral angles for the edges whose identification produces this cell.

Cone-manifolds

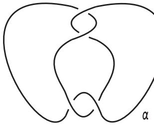
A point in the singular set with conical angle α has a neighborhood isometric to a neighborhood of a point lying on the edge of a wedge with opening angle α whose sides are pairwise identified by way of rotating the 3-space about the edge of the wedge. We can visualise a cone-manifold as a 3-manifold with an embedded graph on which the metric is distorted. Furthermore, if we measure the length of an infinitesimal circle around a component of the graph then instead of the standard $2\pi\varepsilon$ we obtain $\alpha\varepsilon$, where α is the conical angle along the component of the graph.

- Borromean Rings cone-manifold and Lambert cube

This construction done by W. Thurston, D. Sullivan and J.M. Montesinos.



- The figure eight knot 4_1



It was shown in Thurston lectures notes that the figure eight compliment $\mathbb{S}^3 \setminus 4_1$ can be obtained by gluing two copies of a regular ideal tetrahedron. Thus, $\mathbb{S}^3 \setminus 4_1$ admits a complete hyperbolic structure. Later, it was discovered by A.C. Kim, H. Helling and J. Mennicke that the *n*-fold cyclic coverings of the 3-sphere branched over 4_1 produce beautiful examples of the hyperbolic Fibonacci manifolds. Their numerous properties were investigated by many authors. 3-dimensional manifold obtained by Dehn surgery on the figure eight compliment were described by W. P. Thurston. The geometrical structures on these manifolds were investigated in Ph.D. thesis by C. Hodgson.

Geometry of two bridge knots and links. 4_1– knot.

The following result takes a place due to Thurston, Kojima, Hilden, Lozano, Montesinos, Rasskazov and Mednykh.

Theorem

A cone-manifold $4_1(\alpha)$ is hyperbolic for $0 \leq \alpha < \alpha_0 = \frac{2\pi}{3}$, Euclidean for $\alpha = \alpha_0$ and spherical for $\alpha_0 < \alpha < 2\pi - \alpha_0$.

Other 5 exotic geometries on the figure eight cone-manifold were described by E. Molnar, J. Szirmai and A. Vesnin.

The volume of the figure eight cone-manifold in the spaces of constant curvature is given by the following theorem.

Theorem (Rasskazov, Mednykh, 2006)

Let $\mathrm{V}(\alpha) = \mathrm{Vol}\ 4_1(\alpha)$ *and* ℓ_α *is the length of singular geodesic of* $4_1(\alpha)$. *Then*

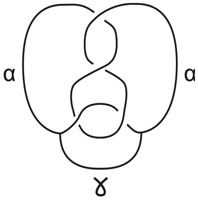
$(\mathbb{H}^3)$ $\mathrm{V}(\alpha) = \int_\alpha^{\alpha_0} \mathrm{arccosh}\,(1 + \cos\theta - \cos 2\theta)\mathrm{d}\theta,\ 0 \leq \alpha < \alpha_0 = \frac{2\pi}{3}$,

$(\mathbb{E}^3)$ $\mathrm{V}(\alpha_0) = \frac{\sqrt{3}}{108}\,\ell_{\alpha_0}^3$,

$(\mathbb{S}^3)$ $\mathrm{V}(\alpha) = \int_{\alpha_0}^{\alpha} \arccos\,(1 + \cos\theta - \cos 2\theta)\mathrm{d}\theta,\ \alpha_0 < \alpha \leq \pi, \quad \mathrm{V}(\pi) = \frac{\pi^2}{5}$,

$\mathrm{V}(\alpha) = 2\mathrm{V}(\pi) - \mathrm{V}(2\pi - \alpha),\ \pi \leq \alpha < 2\pi - \alpha_0.$

We study a geometric structure on the cone-manifold $4_1(\alpha, \alpha; \gamma)$ whose underlying space is the three-dimensional sphere $\mathbb{S}^3$ and the singular set Σ is the figure-eight knot with additional bridge.



Using Wirtinger's algorithm, one can find the fundamental group $\pi_1(\mathbb{S}^3 \setminus \Sigma)$. We find a representation of its generators by rotation matrices in $\mathbb{E}^3$ or $\mathbb{H}^3$. This allows to construct "butterfly" polyhedron in $\mathbb{E}^3$ or $\mathbb{H}^3$ as a fundamental polyhedron for the cone-manifold $4_1(\alpha, \alpha; \gamma)$. We obtain sufficient conditions for the existence of Euclidean or hyperbolic structure on $4_1(\alpha, \alpha; \gamma)$.

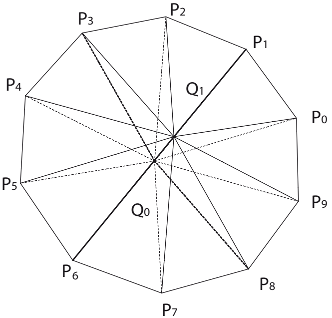
Consider the holonomy mapping $\varphi : \pi_1(\mathbb{S}^3 \setminus \Sigma) \to \mathrm{Isom}(\mathbb{E}^3)$ carrying the generators s and t of the fundamental group $\pi_1(\mathbb{S}^3 \setminus \Sigma) = \langle s, t, \ell : s\ell = \ell s, \ell = s\,t\,s\,t^{-1}s^{-1}t\,s\,t\,s^{-1}t^{-1} \rangle$ to the linear transformations $\mathcal{S} = (x - e_3)S + e_3, \mathcal{T} = (x + e_3)T - e_3$, where $e_3 = (0, 0, 1)$ while S, T are rotation matrices

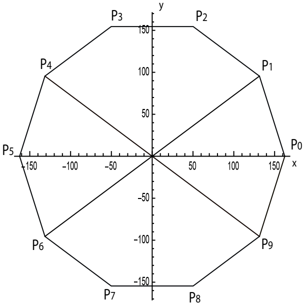
$$S = \frac{1}{M^2+1}\begin{pmatrix} M^2 + \cos\theta & \sin\theta & -2M\sin\frac{\theta}{2} \\ \sin\theta & M^2 - \cos\theta & 2M\cos\frac{\theta}{2} \\ 2M\sin\frac{\theta}{2} & -2M\cos\frac{\theta}{2} & -1 + M^2 \end{pmatrix},$$

$$T = \frac{1}{M^2+1}\begin{pmatrix} M^2 + \cos\theta & -\sin\theta & -2M\sin\frac{\theta}{2} \\ -\sin\theta & M^2 - \cos\theta & -2M\cos\frac{\theta}{2} \\ 2M\sin\frac{\theta}{2} & 2M\cos\frac{\theta}{2} & -1 + M^2 \end{pmatrix}$$

where $M = \cot\frac{\alpha}{2}$ and θ is the angle of relative rotation between singular components. The holonomy mapping carries the element ℓ into the rotation through angle γ about the singular component corresponding to the bridge of the knot. Refer as the *holonomy group* of the manifold $4_1(\alpha, \alpha; \gamma)$ to the group generated by the rotations $\mathcal{S}, \mathcal{T}$ through angle α about the singular component of the fundamental set.

Fundamental polyhedron for the cone-manifold $4_1(\alpha, \alpha; \gamma)$





Fundamental polyhedron $\mathcal{F}$ can be realized in $\mathbb{E}^3$, $\mathbb{H}^3$ and $\mathbb{S}^3$. Identify the curvilinear facets of $\mathcal{F}$ via isometric transformations $\mathcal{S}$ and $\mathcal{T}$ using the rule

$$\mathcal{S} : P_1 P_0 P_9 P_8 P_7 P_6 \to P_1 P_2 P_3 P_4 P_5 P_6,$$

$$\mathcal{T} : P_4 P_5 P_6 P_7 P_8 P_9 \to P_4 P_3 P_2 P_1 P_0 P_9.$$

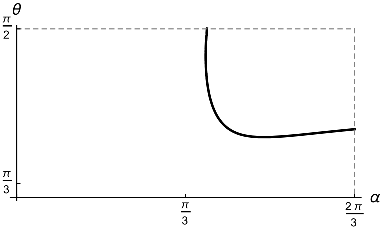
Euclidean structure on $4_1(\alpha, \alpha; \gamma)$

Theorem (Abr., Mednykh, Sokolova)

The cone manifold $4_1(\alpha, \alpha; \gamma)$ *has an Euclidean structure if*

$$5 + 6\,M^2 + M^4 - 60\,X^2 - 12\,M^2X^2 + 80\,X^4 = 0,$$

where $M = \cot\frac{\alpha}{2}, \alpha \in (\frac{\pi}{3}, \frac{2\pi}{3}), X = \cos\frac{\theta}{2}, \theta \in (0, \frac{\pi}{2})$ *and* θ *is the angle of relative rotation between singular components.*



In particular, $4_1(\alpha, \alpha; 2\pi) = 4_1(\alpha)$ is a Euclidean cone-manifold whose singular set is the figure-eight knot with conical angle α (the bridge disappears and we get the situation which was previously known).

Euclidean structure on $4_1(\alpha, \alpha; \gamma)$

Theorem (Abr., Mednykh, Sokolova)

If cone-manifold $4_1(\alpha, \alpha; \gamma)$ *admits an Euclidean structure then its normalised volume*

$$Vol(4_1(\alpha, \alpha; \gamma)) = \frac{8X\sqrt{1 - X^2}(M^4 - 50M^2X^2 + 150X^2 - 25)}{3M^2(1 + M^2 - 8X^2)^2}.$$

Caley-Klein model of hyperbolic 3-space

Consider *Minkowski space* R^4_1 with *Lorentz scalar product*

$$\langle X, Y \rangle = -x_1\, y_1 - x_2\, y_2 - x_3\, y_3 + x_4\, y_4. \qquad (1)$$

The *Caley-Klein model* of hyperbolic space is the set of vectors $K = \{(x_1, x_2, x_3, 1) \ : \ x_1^2 + x_2^2 + x_3^2 < 1\}$ forming the unit 3-ball in the hyperplane $x_4 = 1$. The lines and planes in K are just the intersections of ball K with Euclidean lines and planes in the hyperplane $x_4 = 1$.

The *distance* between vectors V and W is defined as

$$\operatorname{ch} \rho\,(V, W) = \frac{\langle V, W \rangle}{\sqrt{\langle V, V \rangle \, \langle W, W \rangle}}. \qquad (2)$$

A plane in K is a set $\mathcal{P} = \{V \in K \ : \ \langle V, N \rangle = 0\}$, where N is a normal vector to the plane $\mathcal{P}$.

Every of four *dihedral angles* between the planes $\mathcal{P}, \mathcal{Q}$ with normal vectors N, M are defined by relation

$$\cos\,(\widehat{\mathcal{P}, \mathcal{Q}}) = \pm \frac{\langle N, M \rangle}{\sqrt{\langle N, N \rangle \, \langle M, M \rangle}}. \qquad (3)$$

We identify the isometry group $\mathrm{Isom}(\mathbb{H}^3)$ with *positive Lorentz group* $\mathbf{PSO}(1,3)$. The group $\mathbf{O}(1,3)$ is the set of 4×4 matrices with real coefficients preserving the quadratic form (1). **S** stands for considering only elements of determinant 1, **P** stands for factoring out the center.
Consider the representation of the fundamental group $\pi_1(\mathbb{S}^3 \setminus \Sigma) = \langle s, t, \ell : s\ell = \ell s, \ell = s\,t\,s\,t^{-1}s^{-1}t\,s\,t\,s^{-1}t^{-1} \rangle$ in $\mathbf{PSO}(1,3)$.
Its generators s, t are the rotation matrices

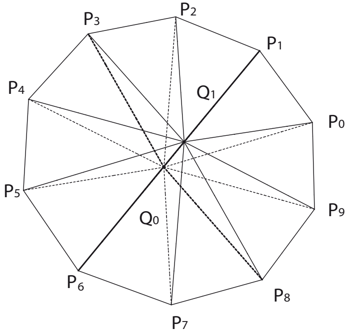
$$S_h = \frac{1}{M^2+1}\begin{pmatrix} M^2+X^2-Y^2 & 2\,X\,Y & -2\,\mathrm{ch}\,h\,M\,Y & -2\,\mathrm{sh}\,h\,M\,Y \\ 2\,X\,Y & M^2-X^2+Y^2 & 2\,\mathrm{ch}\,h\,M\,X & 2\,\mathrm{sh}\,h\,M\,X \\ 2\,\mathrm{ch}\,h\,M\,Y & -2\,\mathrm{ch}\,h\,M\,X & M^2-\mathrm{ch}^2\,h-\mathrm{sh}^2\,h & -2\,\mathrm{ch}\,h\,\mathrm{sh}\,h \\ -2\,\mathrm{sh}\,h\,M\,Y & 2\,\mathrm{sh}\,h\,M\,X & 2\,\mathrm{ch}\,h\,\mathrm{sh}\,h & M^2+\mathrm{ch}^2\,h+\mathrm{sh}^2\,h \end{pmatrix},$$

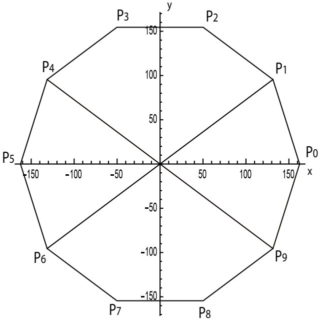
$$T_h = \frac{1}{M^2+1}\begin{pmatrix} M^2+X^2-Y^2 & -2\,X\,Y & -2\,\mathrm{ch}\,h\,M\,Y & 2\,\mathrm{sh}\,h\,M\,Y \\ -2\,X\,Y & M^2-X^2+Y^2 & -2\,\mathrm{ch}\,h\,M\,X & 2\,\mathrm{sh}\,h\,M\,X \\ 2\,\mathrm{ch}\,h\,M\,Y & 2\,\mathrm{ch}\,h\,M\,X & M^2-\mathrm{ch}^2\,h-\mathrm{sh}^2\,h & 2\,\mathrm{ch}\,h\,\mathrm{sh}\,h \\ 2\,\mathrm{sh}\,h\,M\,Y & 2\,\mathrm{sh}\,h\,M\,X & -2\,\mathrm{ch}\,h\,\mathrm{sh}\,h & M^2+\mathrm{ch}^2\,h+\mathrm{sh}^2\,h \end{pmatrix},$$

where $M = \cot\frac{\alpha}{2}, X = \cos\frac{\theta}{2}, Y = \sin\frac{\theta}{2}$.

The coordinates of the vertices of fundamental polyhedron $\mathcal{F}$ in Caley-Klein model of $\mathbb{H}^3$ are as follows

$$P_0 = (x, 0, 0, 1), \qquad P_1 = (tX, tY, \operatorname{th} h, 1),$$
$$P_2 = (a, b, c, 1), \qquad P_3 = (-a, b, -c, 1),$$
$$P_4 = (-tX, tY, -\operatorname{th} h, 1), \qquad P_5 = (-x, 0, 0, 1),$$
$$P_6 = (-tX, -tY, \operatorname{th} h, 1), \qquad P_7 = (-a, -b, c, 1),$$
$$P_8 = (a, -b, -c, 1), \qquad P_9 = (tX, -tY, -\operatorname{th} h, 1),$$
$$Q_0 = (0, 0, \operatorname{th} h, 1), \qquad Q_1 = (0, 0, -\operatorname{th} h, 1).$$





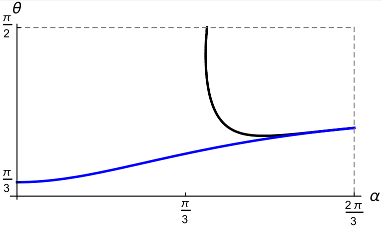
Hyperbolic structure on $4_1(\alpha, \alpha; \gamma)$

Theorem (Abr., Mednykh, Sokolova)

The cone manifold $4_1(\alpha, \alpha; \gamma)$ has a hyperbolic structure if

$$\begin{cases} -1 + 3\,M^2 + 12\,X^2 - 4\,M^2X^2 - 16\,X^4 & \geq 0, \quad (i) \\ 5 + 6\,M^2 + M^4 - 60\,X^2 - 12\,M^2X^2 + 80\,X^4 & > 0, \quad (ii) \end{cases}$$

where $M = \cot\frac{\alpha}{2}, \alpha \in (\frac{\pi}{3}, \frac{2\pi}{3}), X = \cos\frac{\theta}{2}, \theta \in (0, \frac{\pi}{2})$ and θ is the angle of relative rotation between singular components.



The equality in (i) is achieved under the condition $\gamma = 2\pi$, i.e. when the bridge disappears. The equality in (ii) is achieved if there exist an Euclidean structure on $4_1(\alpha, \alpha; \gamma)$.

Natural isomorphism $\mathbf{PSO}(1,3) \cong \mathbf{PSL}(2,\mathbb{C})$

Now we identify the isometry group $\mathrm{Isom}(\mathbb{H}^3)$ with *projective special linear group* $\mathbf{PSL}(2,\mathbb{C})$. The group $\mathbf{PSL}(2,\mathbb{C})$ is the automorphism group of the Riemann sphere. Viewing the Riemann sphere as $\mathbb{C} \cup \{\infty\}$, its automorphisms are given as fractional linear transformations

$$z \mapsto \frac{az+b}{cz+d}, \quad a,b,c,d \in \mathbb{C}, \quad ad-bc \neq 0.$$

The composition of these works like multiplication of the corresponding matrices

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Matrices that are scalar multiples of each other define the same fractional linear transformation, so we need to quotient out by the center.

Hyperbolic structure on $4_1(\alpha, \alpha; \gamma)$

Consider the representation of the fundamental group $\pi_1(\mathbb{S}^3 \setminus \Sigma) = \langle s, t, \ell : s\ell = \ell s, \ell = s\,t\,s\,t^{-1}s^{-1}t\,s\,t\,s^{-1}t^{-1} \rangle$ in $\mathbf{PSL}(2, \mathbb{C})$. The generators s, t are the rotation matrices

$$A = \begin{pmatrix} \cos\alpha & i\,e^{\rho/2}\sin\alpha \\ i\,e^{-\rho/2}\sin\alpha & \cos\alpha \end{pmatrix}, \quad B = \begin{pmatrix} \cos\beta & i\,e^{-\rho/2}\sin\beta \\ i\,e^{\rho/2}\sin\beta & \cos\beta \end{pmatrix}.$$

We put $\alpha = \beta$ and find that $-\cos\frac{\gamma}{2} = \frac{1}{2}\,\mathrm{tr}(A\,B\,A\,B^{-1}A^{-1}B\,A\,B\,A^{-1}B^{-1})$.

Theorem (Fricke)

Let w be a word composed by the product of finitely many 2×2 matrices A, B and their inverses $(\det A = \det B = 1)$. Then there exist a polynomial $P(x, y, z)$ with integer coefficients such that $\mathrm{tr}\,w = P(\mathrm{tr}A, \mathrm{tr}B, \mathrm{tr}(AB))$. This is known as a Fricke polynomial.

This allowed us to prove the next theorem.

Hyperbolic structure on $4_1(\alpha, \alpha; \gamma)$

Theorem (Abr., Mednykh, Sokolova)

If cone-manifold $4_1(\alpha, \alpha; \gamma)$ admits a hyperbolic structure then

$$-\cos\frac{\gamma}{2} = 8\,u^2 - 16\,u^4 + 5\,w - 40\,u^2 w + 80\,u^4 w + 32\,u^2 w^2 - 128\,u^4 w^2 - 20\,w^3 + 64\,u^2 w^3 + 64\,u^4 w^3 - 64\,u^2 w^4 + 16\,w^5,$$

where $u = \frac{1}{2}\,\mathrm{tr}A = \frac{1}{2}\,\mathrm{tr}B = \cos\alpha,\ w = \mathrm{tr}(A\,B^{-1}) = u^2 - (1-u^2)\,\mathrm{ch}\,\rho$ *and* ρ *is the complex hyperbolic distance between the singular components of* $4_1(\alpha, \alpha; \gamma)$.

This allows to find the complex hyperbolic distance ρ between the singular components of the cone-manifold $4_1(\alpha, \alpha; \gamma)$ with given conical angles α, γ.

Thank you for attention!