Heavy subgraphs for Hamiltonian properties

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Forbidden subgraphs

Let G be a graph. A subgraph G' of G is an induced subgraph if G' contains all edges $uv \in E(G)$ with $u, v \in V(G')$.

For a given graph H, G is H-free if G contains no induced subgraph isomorphic to H.

For a class $\mathcal{H}$ of graphs, G is $\mathcal{H}$-free if G is H-free for every $H \in \mathcal{H}$.

If G is H-free, then H is called a forbidden subgraph of G.

Forbidden subgraph conditions for hamiltonicity

Theorem (Duffus, Jacboson, Gould, 1981)
Let G be a $\{K_{1,3}, N\}$-free graph.

- *If G is connected, then G is traceable.*
- *If G is 2-connected, then G is hamiltonian.*

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- (Broersma, Veldman, 1990)
If G is $\{K_{1,3}, P_7, D\}$-free, then G is hamiltonian.
- (Faudree, Ryjáček, Schiermeyer, 1995)
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Theorem (Bedrossian, 1991)
Let R and S be connected graphs of order at least 3 with $R, S \neq P_3$, and let G be a 2-connected graph. Then G being $\{R, S\}$-free implies that G is hamiltonian if and only if (up to symmetry) $R = K_{1,3}$ and $S = P_4, P_5, P_6, C_3, Z_1, Z_2, B, N$ or W.

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A graph G is claw-heavy if every induced claw of G has two end-vertices with degree sum at least $|V(G)|$.

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Heavy subgraphs

We generalized the terminology of claw-free to other subgraphs.

Let G be a graph of order n, and G' be an induced subgraph of G. We say G' is a heavy subgraph of G if there are two nonadjacent vertices in $V(G')$ with degree sum at least n in G.

For a given graph H, G is H-heavy if every induced subgraph of G isomorphic to H is heavy. For a class of graphs $\mathcal{H}$, G is $\mathcal{H}$-heavy if G is H-heavy for every $H \in \mathcal{H}$.

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Note that if G is H-free, then G is H-heavy; and if H_1 is an induced subgraph of H_2, then an H_1-heavy graph is also H_2-heavy.

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Comparing with Bedrossian's theorem, we can see that the only graphs for S appearing in Bedrossian's theorem but not here is P_6.

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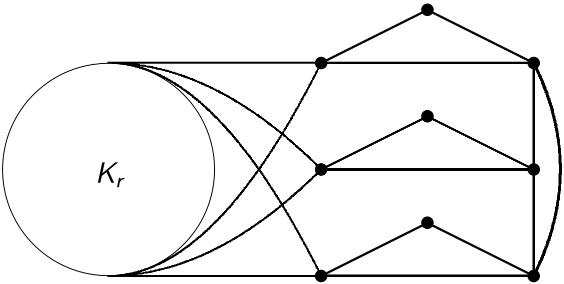
A 2-connected $\{K_{1,3}, P_6\}$-heavy non-hamiltonian graph



A 2-connected $\{K_{1,3}, P_6\}$-heavy non-hamiltonian graph ($r \geq 5$).

A 2-connected $\{K_{1,3}, Z_3\}$-heavy non-hamiltonian graph

It is known that every 2-connected $\{K_{1,3}, Z_3\}$-free graph of order $n >= 10$ is also Hamiltonian (Faudree, Gould, Ryjacek, Schiermeyer, 1995), But this is not true for $\{K_{1,3}, Z_3\}$-heavy graphs.

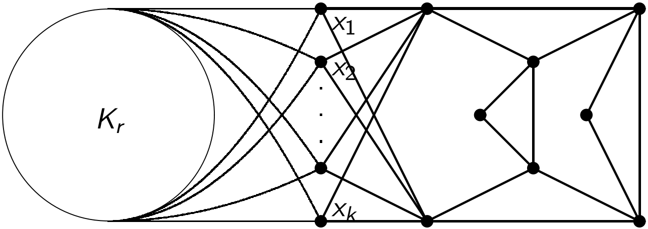


Fig. 4. A 2-connected $\{K_{1,3}, Z_3\}$-heavy non-Hamiltonian graph ($k \geq 7$, $r \geq k + 4$).

Hamiltonian properties

By hamiltonian properties, we mean properties that implying hamiltonicity or implied by hamiltonicity. (Traceability, hamiltonicity, pancyclicity, Hamilton-connectedness,...)

Traceability

Each property has a necessary connectivity. Since every traceable graph is connected, we say that the necessary connectivity of traceability is 1, (and the necessary connectivity of hamiltonicity is 2,) and we consider the class of connected graphs for traceability.

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Necessary degree sum index; o_k-heavy subgraph.

Another remark concerns the degree conditions we impose on certain non-adjacent vertices. When we consider a hamiltonian property P, it is always easy to construct a graph with a large minimum degree that does not satisfy the property P. For instance, the complete bipartite graph $K_{n/2-1,n/2+1}$ is not traceable, and every two nonadjacent vertices of it have degree sum at least $n-2$. On the other hand, a counterpart of Ore's Theorem shows that every graph on n vertices in which every pair of nonadjacent vertices has degree sum at least $n-1$, is traceable. So we call $n-1$ the necessary degree sum index for traceability.

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For a graph G of order n and an induced subgraph G' of G, G' is o_k-heavy if there are two nonadjacent vertices in $V(G')$ with degree sum at least $n+k$ in G. For a given graph H, G is H-o_k-heavy if every induced subgraph of G isomorphic to H is o_k-heavy.
So H-heavy $\iff$ H-o_0-heavy.

o_{-1}-heavy subgraph conditions for traceability

For o_{-1}-heavy subgraph conditions, perhaps surprisingly there exists only one pair for the property of traceability.

Theorem (Li, Zhang, 2015+)
Let R and S be connected graphs of order at least 3 with $R, S \neq P_3$ and let G be a connected graph. Then G being $\{R, S\}$-o_{-1}-heavy implies G is traceable if and only if (up to symmetry) $R = K_{1,3}$ and $S = C_3$.

Block-chains

Note that C_3-o_{-1}-heavy is in fact equivalent to C_3-free. In order to obtain better results, it was observed that many graphs that were used to prove the 'only-if' part of the above theorem were almost trivially non-traceable, in the sense that they contain at least three end blocks. To exclude such graphs, we turned to block-chains.

A block-chain is a graph whose block graph is a path, i.e., it is either non-separable or has exactly two end-blocks.

Subgraph conditions for traceability of block-chains

Theorem (Li, Broersma, Zhang, 2013)
Let R and S be connected graphs of order at least 3 with $R, S \neq P_3$ and let G be a block-chain. Then G being $\{R, S\}$-free implies G is traceable if and only if (up to symmetry) $R = K_{1,3}$ and S is an induced subgraph of $N_{1,1,3}$, or $R = K_{1,4}$ and $S = P_4$.

Theorem (Li, Broersma, Zhang, 2014)
Let R and S be connected graphs of order at least 3 with $R, S \neq P_3$ and let G be a block-chain. Then G being $\{R, S\}$-o_{-1}-heavy implies G is traceable if and only if (up to symmetry) $R = K_{1,3}$ and S is an induced subgraph of N or W.

Homogeneously traceable graphs

A graph is homogeneously traceable if it contains a Hamilton path starting from any vertex. So the necessary connectivity of homogeneous traceability (for graphs of order at least 3) is 2; and Since $K_{(n-1)/2,(n+1)/2}$ is not homogeneously traceable, which means the necessary degree sum index is n.

Theorem (Li, Broersma, Zhang, 2013)
Let R and S be connected graphs of order at least 3 with $R, S \neq P_3$ and let G be a 2-connected graph. Then G being $\{R, S\}$-free implies G is homogeneously traceable if and only if (up to symmetry) $R = K_{1,3}$ and S is an induced subgraph of $B_{1,4}, B_{2,3}$ or $N_{1,1,3}$.

Theorem (Trivial)
Let R and S be connected graphs of order at least 3 with $R, S \neq P_3$ and let G be a 2-connected graph. Then G being $\{R, S\}$-heavy implies G is homogeneously traceable if and only if (up to symmetry) $R = K_{1,3}$ and S is an induced subgraph of N or W.

Subgraph conditions for pancyclicity

Necessary connectivity: 2.
Necessary degree sum index: $n + 1$.

Theorem (Bedrossian, 1991)
Let R and S be connected graphs of order at least 3 with $R, S \neq P_3$ and let G be a 2-connected graph which is not a cycle. Then G being $\{R, S\}$-free implies G is pancyclic if and only if (up to symmetry) $R = K_{1,3}$ and S is an induced subgraph of P_5 or Z_2.

Theorem (Li, Ning, Broersma, Zhang, 2015+)
Let R and S be connected graphs of order at least 3 with $R, S \neq P_3$ and let G be a 2-connected graph which is not a cycle. Then G being $\{R, S\}$-o_1-heavy implies G is pancyclic if and only if (up to symmetry) $R = K_{1,3}$ and S is an induced subgraph of P_5 or Z_2.

Subgraph conditions for 2-factors

Necessary connectivity: –.
Necessary degree sum index: n.

Theorem (Faudree, Faudree, Ryjáček, 2008)
Let R and S be connected graphs of order at least 3 with $R, S \neq P_3$ and let G be a 2-connected graph of order at least 10. Then G being $\{R, S\}$-free implies G has a 2-factor if and only if (up to symmetry) $R = K_{1,3}$ and S is an induced subgraph of $B_{1,4}$ or $N_{1,1,3}$, or $R = K_{1,4}$ and $S = P_4$.

Theorem (Li, Ryjáček, Yoshimoto, 2015+)
Let R and S be connected graphs of order at least 3 with $R, S \neq P_3$ and let G be a 2-connected graph of order at least 10. Then G being $\{R, S\}$-heavy implies G has a 2-factor if and only if (up to symmetry) $R = K_{1,3}$ and S is an induced subgraph of $N_{1,1,3}$.

Hamilton-connectedness

Necessary connectivity: 3.
Necessary degree sum index: $n + 1$.

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Thank you for attention!