Supereulerian graphs and hamiltonian line graphs

Hong-Jian Lai, West Virginia University

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The Supereulerian Problem

• Boesch, Suffel and Tindel (JGT, 1977) proposed the supereulerian problem, which seeks a characterization of graphs that have spanning Eulerian subgraphs.

• A graph G containing a spanning eulerian subgraph (that is, a spanning closed trail) is a supereulerian graph.

• Theorem (Pulleyblank, JGT, 1979) proved that determining whether a graph is supereulerian, even within planar graphs, is NP-complete.

Hamiltonian Line Graphs

• Theorem (Harary and Nash-Williams, Canad. Math. Bull. 1965) For G with $|E(G)| \geq 3$, the line graph $L(G)$ is hamiltonian iff G has an eulerian subgraph H with $E(G - V(H)) = \emptyset$, (Called dominating closed trail, DCT, of G).

• If G is supereulerian, then $L(G)$ is hamiltonian.

• Theorem (Ryjáček, JCTB, 1997) Hamiltonian claw-free graph problem can be converted into the hamiltonian line graph problem.

Catlin's Reduction and Collapsible Graphs

- Catlin (JGT 1988) wanted to determine graphs H such that for any graph G that contains H as a subgraph, G/H is supereulerian iff G is supereulerian.

- Catlin defined: A graph H is collapsible if for any subset $R \subseteq V(H)$ with $|R|$ being even, H has a spanning, connected subgraph $\Gamma(R)$ such that the set of odd degree vertices of $\Gamma(R)$ is R.

Catlin's Reduction and Collapsible Graphs

- Theorem (Catlin, JGT 1988) If H is a collapsible subgraph of G, then G/H is supereulerian iff G is supereulerian.

- Catlin's reduction method:

Given a graph G, one can repeatedly contract nontrivial collapsible subgraphs until none left. The resulting graph G' is called the reduction of G. Catlin showed that G is supereulerian if and only if G' is supereulerian.

This method is often applied in an inductive argument to show that a graph G has an eulerian subgraph with certain given properties, such as spanning or dominating.

Catlin's Reduction and Collapsible Graphs

• Catlin defined the **kernel** of supereulerian graphs as the family $\mathcal{S}^o = \{H : \forall G \supseteq H, G$ is supereulerian iff G/H is supereulerian $\}$.

• Catlin (JGT Survey, 1992) conjecture: A graph H is collapsible if and only if G is in $\mathcal{S}^o$.

Catlin's Reduction and Collapsible Graphs

• Theorem: A graph H is collapsible if and only if for any graph G containing H as an induced subgraph, both of the following hold:
(i) If G has an eulerian subgraph Γ such that $V(\Gamma) \cap V(H) \neq \emptyset$, then $(\Gamma \cup H)/H$ is an eulerian subgraph Γ' satisfying $E(\Gamma') = E(\Gamma) - E(H)$; and
(ii) If Γ' is an eulerian subgraph in G/H containing v_H, the vertex in G/H onto which H is contracted, then G has an eulerian subgraph Γ such that $V(H) \subseteq V(\Gamma)$ and $E(\Gamma') = E(\Gamma) - E(H)$.

Bauer's Problem

• D. Bauer (1985) proposed to determine the best possible constant c such that every simple, connected graph G on n vertices with $\delta(G) \geq cn$ will have a hamiltonian line graph.

• Using his reduction method, Catlin (JGT 1988) settled Bauer's problem and showed that if $\delta(G) > \frac{n}{5} - 1$, then for sufficiently large n, G is supereulerian. An infinite family of graphs contractible to $K_{2,3}$ indicates that this bound is best possible.

Bauer's Problem

• Catlin and Z. H. Chen: proved that for any real numbers a and b with $0 < a < 1$, there exists a finite family $\mathcal{F}$ of non supereulerian graphs and an integer $N = N(a, b)$ such that every simple graph on $n \geq N$ vertices with if $\delta(G) \geq an + b$ is supereulerian iff G cannot be contracted to a member in the family $\mathcal{F}$.

Kuipers and Veldman Conjecture

- Kuipers and Veldman conjectured that any 3-connected claw-free graph with order n and minimum degree $\delta \geq \frac{n+6}{10}$ is Hamiltonian for n sufficiently large.

- Theorem (Kuipers and Veldman, 1998) If H is a 3-connected claw-free simple graph with sufficiently large order n, and if $\delta(H) \geq \frac{n+29}{8}$, then H is hamiltonian.

Kuipers and Veldman Conjecture

• Theorem (Favaron and Fraisse, JCTB 2001) If H is a 3-connected claw-free simple graph with order n, and if $\delta(H) \geq \frac{n+37}{10}$, then H is hamiltonian.

• Theorem (Y Shao, M. Zhan, HJL, JCTB 2006) If H is a 3-connected claw-free simple graph with $n \geq 196$, and if $\delta(H) \geq \frac{n+5}{10}$, then either H is hamiltonian, or $\delta(H) = \frac{n+5}{10}$ and $cl(H)$ is the line graph of G obtained from the Petersen graph P_{10} by adding $\frac{n-15}{10}$ pendant edges at each vertex of P_{10}.

• Consider the exceptional graph is "generated by the Petersen graph P_{10}".

Kuipers and Veldman Conjecture

• Theorem (Z.-H. Chen, L. Xiong and HJL) For any real numbers a and b with $0 < a < 1$, there exist an integer $N = N(a, b)$ and a "finitely generated" family $\mathcal{F}$ of graphs such that every 3-connected claw-free simple graph G with $n \geq N$ vertices and with $\delta(H) \geq an+b$ is hamiltonian iff the closure $cl(H) = L(G)$ for a graph $G \notin \mathcal{F}$.

Hamiltonian-connected line graphs and spanning trailable graphs

- For $e, e' \in E(G)$, an (e, e')-trail is a trail of G having the end-edges e and e'.

- An (e, e')-trail is dominating if each edge of G is incident with at least one internal vertex of the trail

- Theorem: $L(G)$ is hamiltonian-connected iff for any pair of edges $e, e' \in E(G)$, G has a dominating (e, e')-trail.

Hamiltonian-connected line graphs and spanning trailable graphs

• An (e, e')-trail is spanning if it is a dominating trail and it contains all the vertices of G.

• Spanning (e, e')-trails are dominating (e, e')-trails.

• A graph G is spanning trailable if for any pair of edges $e, e' \in E(G)$, G has a spanning (e, e')-trail.

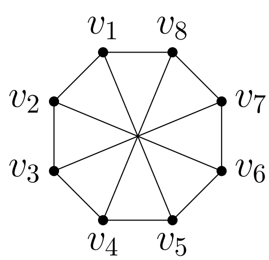
• Theorem: If G is spanning trailable, then $L(G)$ is hamiltonian-connected.

Bauer's Problem, hamiltonian-connected version

• Theorem (Jianping Liu, Aimei Yu, Keke Wang) Let G be a connected simple graph on n vertices and let a, b be real numbers with $0 < a \leq 1$. There exist an integer $N = N(a, b)$ and a finite family $\mathcal{F} = \mathcal{F}(a, b)$ such that if $n \geq N$ and if for any $u, v \in V(G)$ with $uv \notin E(G)$, $d_G(u) + d_G(v) \geq a|V(G)| + b$ then exactly one of the following must hold:
(i) $L(G)$ is hamiltonian-connected;
(ii) $\kappa(L(G)) \leq 2$;
(iii) G can be contracted to a member in $\mathcal{F}$.

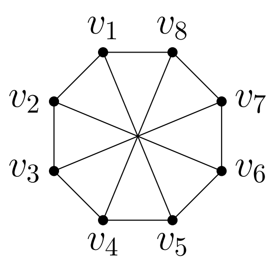
Bauer's Problem, hamiltonian-connected version

• Theorem (Jianping Liu, Aimei Yu, Keke Wang) Let G be a connected simple graph on n vertices. If for any $u, v \in V(G)$ with $uv \notin E(G)$, $d_G(u) + d_G(v) \geq \frac{n}{4} - 2$, then for sufficiently large n, exactly one of the following must hold:
(i) $L(G)$ is hamiltonian-connected;
(ii) $\kappa(L(G)) \leq 2$;
(iii) G can be contracted to W_8.



Bauer's Problem, hamiltonian-connected version

• Theorem (Jianping Liu, Aimei Yu, Keke Wang) Let G be a connected simple triangle free graph on n vertices. If for any $u, v \in V(G)$ with $uv \notin E(G)$, $d_G(u) + d_G(v) \geq \frac{n}{8}$, then for sufficiently large n, exactly one of the following must hold:
(i) $L(G)$ is hamiltonian-connected;
(ii) $\kappa(L(G)) \leq 2$;
(iii) G can be contracted to W_8.



Kuipers and Veldman Conjecture: Hamiltonian connected version

• The closures allow us to focus on line graphs.

• Theorem (M. Zhan and HJL) Let $H = L(G)$ be a 3-connected line graph on n vertices. For sufficiently large n, if $\delta(H) \geq \frac{n+4}{8}$, then either H is hamiltonian-connected or equality holds and G is obtained from W_8 by attaching exactly $(n-12)/8$ pendant edges to every vertex of W_8 ("generated by W_8").

• Theorem (M. Zhan and HJL) For any real numbers a and b with $0 < a < 1$, there exist an integer $N = N(a, b)$ and a "finitely generated" family $\mathcal{F} = \mathcal{F}(a, b)$ such that if $H = L(G)$ be a 3-connected line graph on $n \geq N$ vertices and if $\delta(H) \geq an+b$, then either H is hamiltonian-connected or equality holds and $G \in \mathcal{F}$.

Supereulerian width and spanning connectivity of line graphs

• For an integer $s > 0$ and for $u, v \in V(G)$ with $u \neq v$, an $(s; u, v)$-trail-system of G is a subgraph H consisting of s edge-disjoint (u, v)-trails.

• G is supereulerian iff for $u, v \in V(G)$, G contains a spanning $(2; u, v)$-trail-system.

Supereulerian width and spanning connectivity of line graphs

• Recall: G is collapsible iff for any even subset $R \subseteq V(G)$, G has a spanning connected subgraph $\Gamma(R)$ such that the odd degree vertices of $\Gamma(R)$ is R.

• $R = \{u, v\}$, $\Gamma(R)$ is a spanning $(1; u, v)$-trail-system.

• $R = \emptyset$, $\Gamma(R)$ is a spanning $(2; u, v)$-trail-system.

• The **supereulerian width** $\mu'(G)$ of a graph G is the largest integer s such that for every integer k with $0 \le k \le s$, and for every pair of vertices $u, v \in V(G)$, G has a spanning $(k; u, v)$-trail-system.

Supereulerian width and spanning connectivity of line graphs

• For an integer $s > 0$ and for $u, v \in V(G)$ with $u \neq v$, an $(s; u, v)$-path-system of G is a subgraph H consisting of s internally disjoint (u, v)-paths.

• G is hamiltonian iff for $u, v \in V(G)$, G contains a spanning $(2; u, v)$-path-system.

• G is hamiltonian-connected iff for $u, v \in V(G)$, G contains a spanning $(1; u, v)$-path-system.

• The **spanning connectivity** $\kappa^*(G)$ of a graph G is the largest integer s such that for every integer k with $0 \leq k \leq s$, and for every pair of vertices $u, v \in V(G)$, G has a spanning $(k; u, v)$-path-system.

Supereulerian width and spanning connectivity of line graphs

• A dominating $(k; e', e'')$-trail systems in G is a subgraph H consisting of k edge-disjoint (e', e'')-trail $(T_1, T_2, \cdots, T_k)$ such that every edge of G is incident with an internal vertex of T_i for some i, $(1 \leq i \leq k)$.

• Theorem (Y. Chen et al, GC 2013) Let $s \geq 1$ be an integer, and G a graph with $|E(G)| \geq 3$. The following are equivalent.
(i) $\kappa^*(L(G)) \geq s$;
(ii) For any edge e', $e'' \in E(G)$, G has a dominating $(k; e', e'')$-trail-system, for all $1 \leq k \leq s$.

Leading Conjectures

- Conjecture (Thomassen) Every 4-connected line graph is hamiltonian.

- Conjecture (Matthews and Sumner) Every 4-connected claw-free graph is hamiltonian.

- Conjecture (Kučzel and Xiong) Every 4-connected line graph is hamiltonian-connected.

- Conjecture (Ryjáček and Vrána) Every 4-connected claw-free graph is hamiltonian-connected.

Leading Conjectures

• Conjecture (Saito) Every 3-connected line graph of diameter at most 3 is hamiltonian unless it is the line graph of a graph obtained from the Petersen graph by adding at least one pendant edge to each of its vertices.

• Conjecture (Hamiltonian-connected version) There exists a "finitely generated" family $\mathcal{F}$ of graphs such that every 3-connected line graph $L(G)$ of diameter at most 3 is hamiltonian-connected unless $G \in \mathcal{F}$.

- **Conjecture** For any integers $r, s > 0$, there exists an integer $k(r, s)$ such that every $k(r, s)$-connected $K_{1,r}$-free graph has spanning connectivity at least s.

- Leading Conjectures: $k(3, 2) = 4$.