On dominating even subgraphs in cubic graphs

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joint work with R. Cada, S. Chiba, K. Ozeki

2004

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Conjecture. (Fujisawa, Xiong, Y, Zhang 2007)

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- A 2-connected claw-free graph with $\delta \geq 4$ has a 2-factor with $\dfrac{n+1}{4}$ components. (*Jackson, Y 2007*)
- A 3-connected claw-free graph with $\delta \geq 4$ has a 2-factor with $\dfrac{2n}{15}$ components. *(Jackson, Y 2009)*

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- A claw-free graph with $\delta \geq 4$ has a 2-factor with at most $\frac{n}{\delta - 1}$ components. (*Broersma, Paulusma, Y 2009*)

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- If G is a line graph with $\delta \geq 7$,
  $\Longrightarrow$ for any independent set S,
  G has a 2-factor such that
  each cycle contains ≤ 1 vertex in S.

2009

<u>**Theorem.**</u> (*Kuzel, Ozeki, Y 2012*)

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- If G is a 3-connected claw-free graph with $\delta \geq 4$,

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- There is an infinite family of essentially 4-edge-connected cubic graphs in which every 2-factor contains a 5-cycle.
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Theorem. (Cada, Chiba, Ozeki, Y 2015+)

- A 3-edge-connected cubic graph has a dominating even subgraph F such that

  each cycle of F contains $\geq$ 6 vertices and

  F intersects all 3-cuts.

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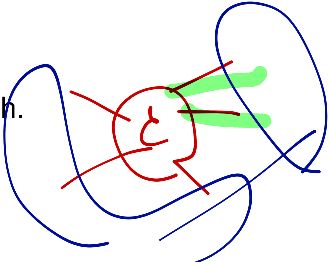

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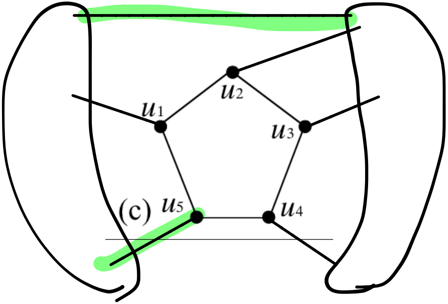

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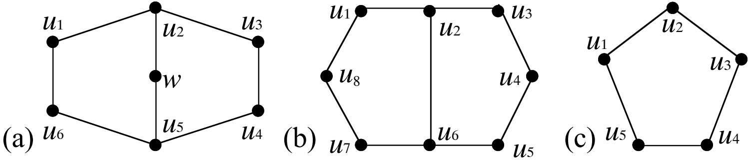

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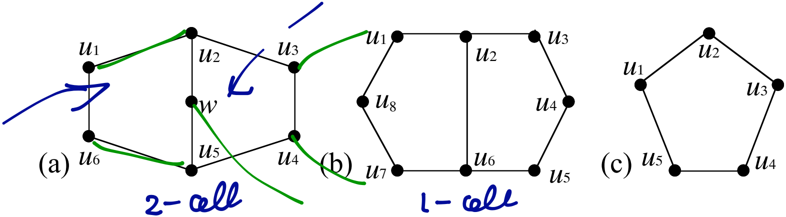

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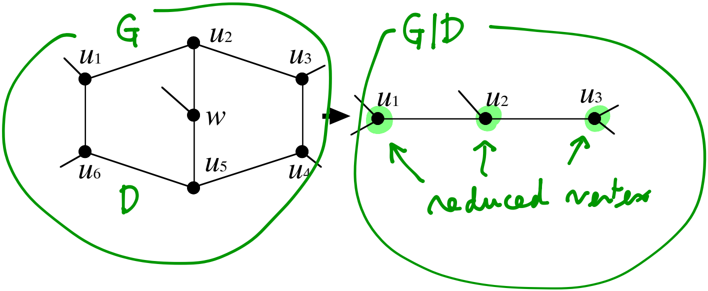

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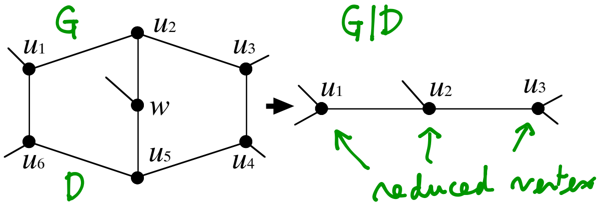

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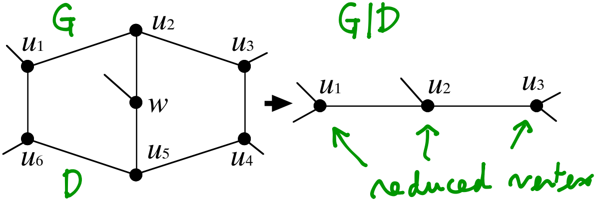

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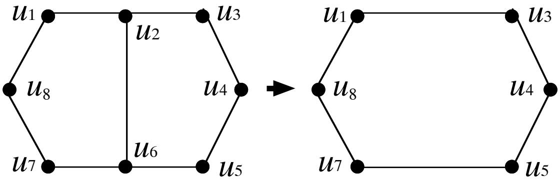

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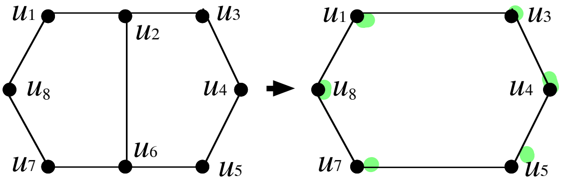

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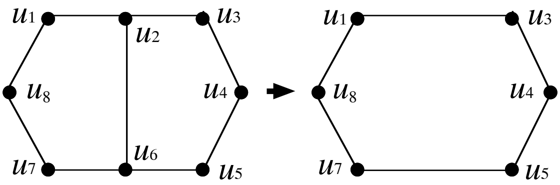

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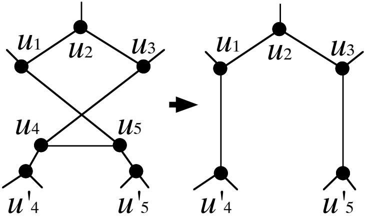

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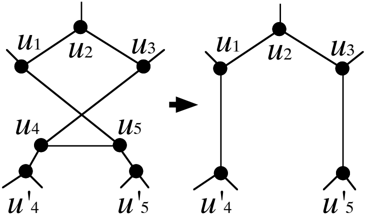

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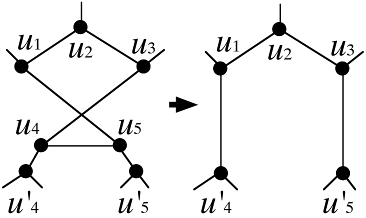

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By the theorem of Kaiser and Skrekovski, we can obtains a 2-factor F' such that

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- if F' contains a 5-cycle C, then C contains a reduced vertex.

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we construct a desired even subgraph F of G.

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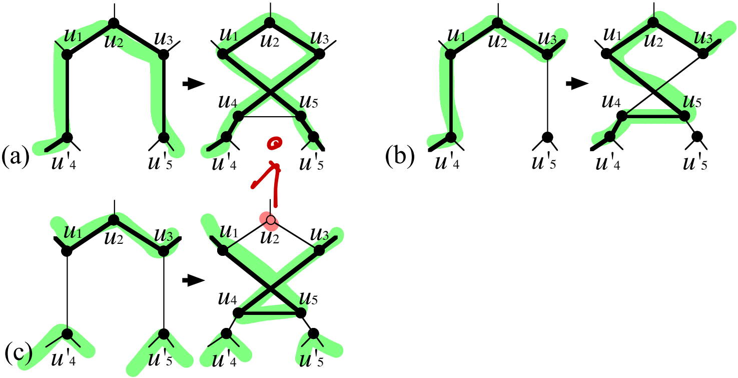

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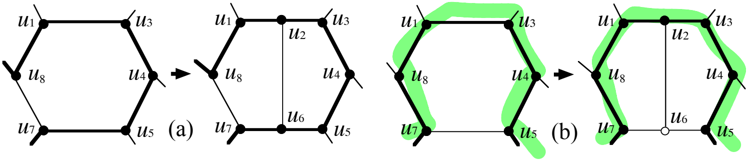

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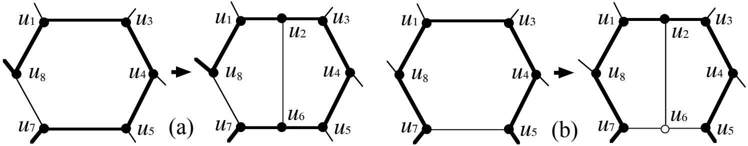

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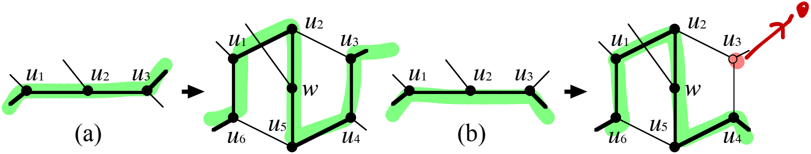

□

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<u>Problem.</u>

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9

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Problem.

Does a 3-edge-connected cubic graph have a dominating even subgraph in which every cycle contains $\geq$ 8 vertices?

Conjecture.

Any 3-edge-connected graph has a dominating even subgraph in which every component contains $\geq$ 6 vertices.

Thank you for your attention.