Hamilton cycles in essentially 9-connected line graphs

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(joint work with **Petr Vrána**)

8th Workshop on the Matthews-Sumner Conjecture, Plzeň, March 30, 2015

Context: A conjecture of Thomassen

Conjecture (Thomassen, 1986)

Every 4-connected line graph is hamiltonian.

- many equivalent forms, e.g.: All 4-connected claw-free graphs are Hamilton-connected

Theorem (K, Vrána 2012)

All 5-connected line graphs with minimum degree ≥ 6 are hamiltonian.

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a vertex cut C in G is essential if at most one component of $G - C$ has edges

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- Yang, Lai, Li and Guo 2012: All 3-connected, essentially 11-connected line graphs are hamiltonian
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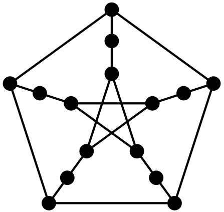
Lower bounds

non-hamiltonian, 3-connected essentially 4-connected graphs are known, e.g., the line graph of



The preimage

let H be a 3-connected, essentially 9-connected line graph, G its preimage: $H = L(G)$

- all edge-cuts of size 1 or 2 in G separate a vertex
- all edge-cuts of size 3 to 8 separate a vertex or an edge

we are looking for a connected eulerian subgraph of H dominating each edge (Harary–Nash-Williams)

Black and white vertices

we choose a maximal independent set W of 3-vertices
W are white vertices, rest are black

a connected subgraph $G' \subseteq G$ is admissible if it covers all black vertices and each white vertex has degree 0 or 2 in G'

ideal situation: an admissible tree with connected complement

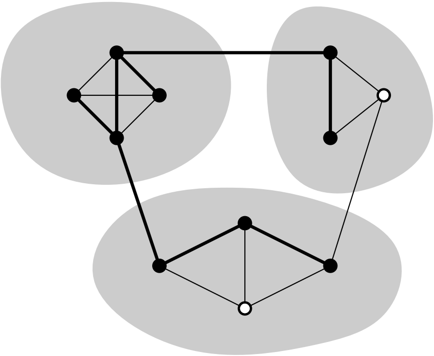
Skeletal Lemma

Lemma

Given G and W, there exists an admissible forest T and a partition $\mathcal{P}$ of $V(G)$ with the following properties:

1. *for $P \in \mathcal{P}$, $T[P]$ is connected except for single white vertices,*
2. *$\overline{T}[P]$ is ('almost') connected for $P \in \mathcal{P}$,*
3. *$\overline{T}/\mathcal{P}$ is acyclic.*

Skeletal partition: example



Counting

so both $T/\mathcal{P}$ and $\overline{T}/\mathcal{P}$ are acyclic

suppose $|\mathcal{P}| > 1$

white vertices not covered by T: leftover vertices

n_0 = number of leftover vertices

$m(T)$, $m(\overline{T})$ = number of edges of T or $\overline{T}$, respectively

$$m(T) \leq n - n_0 - 1$$
$$m(\overline{T}) \leq n - 1$$

Summing we obtain:

$$\sum (deg(v) - 4) + 2n_0 \leq -4$$

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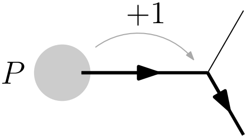
Charges

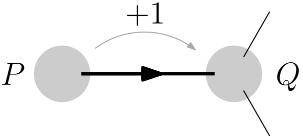
we assign the following charges to vertices:

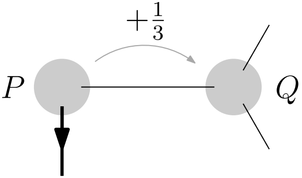
leftover vertices	$+1$
other vertices	$deg(v) - 4$

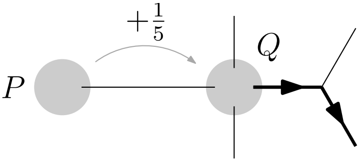
total charge is negative

Rules





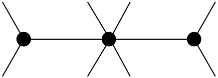


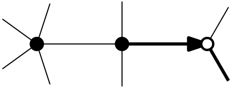


Resulting charge

after redistribution, all vertices have nonnegative charge

proof based on forbidden configurations such as





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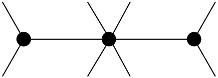

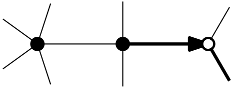

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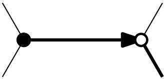

Contradiction

the contradiction shows that $|\mathcal{P}| = 1$ so T is a tree and $\overline{T}$ is 'almost' connected

we augment T to a connected eulerian subgraph F using edge-disjoint paths from $\overline{T}$

F covers all except possibly some of W $\rightarrow$ dominates each edge as W is independent

thus $L(G)$ is hamiltonian

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Extensions

The result can be extended to:

- claw-free graphs (by a closure technique due to Ryjáček and Vrána)
- Hamilton-connectedness

Thank you for your attention.