Hourglasses and Hamilton cycles in 4-connected claw-free graphs

Tomáš Kaiser
MingChu Li
Zdeněk Ryjáček
Liming Xiong

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Observation [Folklore]. *Every 4-connected claw-free and hourglass-free graph is hamiltonian.*

A graph G has the *hourglass property* if in every induced hourglass S, there are two non-adjacent vertices which have a common neighbor in $G - V(S)$.

Theorem 1. *Every 4-connected claw-free graph with the hourglass property is hamiltonian.*

1 Stability

Proposition 2. *Let G be a claw-free graph having the hourglass property. Then $cl(G)$ has the hourglass property.*

An induced hourglass in G is *permanent* if its vertex set induces an hourglass in $cl(G)$.

A graph G has the *permanent hourglass property* if for every permanent induced hourglass $S \subset G$, some two non-adjacent vertices of S have a common neighbor in $G - V(S)$.

(The permanent hourglass property is just the hourglass property restricted to permanent hourglasses.)

Proposition 3. *Let G be a claw-free graph and let $x \in V(G)$ be a locally connected vertex. If G has the permanent hourglass property, its local completion G'_x at x has the permanent hourglass property.*

2 Collapsible graphs

Catlin: A graph G is *collapsible* if for any subset $X \subset V(G)$ of even size, one can find a connected spanning subgraph $H \subset G$ such that the set of vertices v with odd degree $d_H(v)$ is precisely X.

If H is any subgraph of G, then the graph G/H is obtained by contracting H to a single vertex, removing all loops but keeping all multiple edges.

Theorem 4 [Catlin]. *Let H be a collapsible subgraph of G. Then G has a spanning closed trail if and only if G/H does.*

Theorem 5 [Catlin]. *Any 4-edge-connected graph is collapsible.*

Theorem 6 [Lai]. *Let G be a 2-connected graph with minimum degree $\delta(G) \geq 3$. If every edge of G is contained in a cycle of length at most 4, then G is collapsible.*

3 Line graphs

An *I-tree* in G is any subgraph of G isomorphic to the (unique) tree J on 6 vertices, 2 of which have degree 3 in J.

A graph G has the *I-tree property* if in any I-tree $J \subset G$, there are two vertices of distance 3 in J that are adjacent in G.

Lemma 7. *A triangle-free graph G has the I-tree property if and only if its line graph $L(G)$ has the hourglass property.*

A closed trail T in a graph G is *dominating* if $G - V(T)$ is an edgeless graph.

Theorem 8 [Harary–Nash-Williams]. *The line graph $L(G)$ of a graph G is hamiltonian if and only if G has a dominating closed trail.*

A graph is *essentially k-edge-connected* if every edge cut of size less than k is trivial (no more than one component contains any edges). It is easy to see that G is essentially k-edge-connected if and only if its line graph $L(G)$ is k-connected.

Proposition 9. *Any essentially 4-edge-connected, triangle-free graph with the I-tree property has a dominating closed trail.*

Proof.

A – the set of vertices of degree 1 in G

$B_\diamond \subset V(G - A)$ – the set of vertices of degree 2 contained in some 4-cycle

B – the set of all other degree 2 vertices of $G - A$

$G_\diamond$ – the graph obtained from $G - A$ by suppressing all vertices in $B_\diamond$

Theorem 6 $\Rightarrow$ each component of $G_\diamond - B$ is collapsible.

(For a component C of $G_\diamond - B$, the I-tree property is needed to verify that $\delta(C) \geq 3$.)

H – the graph obtained from $G_\diamond$ by suppressing all vertices in B

Consider all non-trivial components C of $G_\diamond - B$ as subgraphs of H, contract them and suppress all vertices in B.

H' – the resulting graph.

Each vertex of the resulting graph H' corresponds to a component of $G' - B$, each edge of H' corresponds to a vertex in B.

H' is 4-edge-connected (since every edge cut in H' corresponds to an *essential* edge cut in G of the same size).

Theorem 5 $\Rightarrow$ H' has a spanning closed trail

Theorem 4 $\Rightarrow$ H has a spanning closed trail T

Straightforward: the corresponding closed trail in G is a dominating closed trail in G.

Proof of Theorem 1: Proposition 3, Theorem 8.

Remark. *Theorem 1 cannot be relaxed to 3-connected graphs, even with a lower bound on minimum degree.*