Problems from the Workshop on Dominating Cycles, Hájek, Czech Republic, April 2003

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This is a list of all problems discussed at the workshop, or at least all of those I heard and haven't forgotten.

Herbert's conjecture

Herbert Fleischner was only present in the form of this conjecture he sent us by e-mail:

Conjecture 1 (Fleischner) *If a 4-connected graph is hamiltonian, then it has at least two Hamilton cycles.*

This is related to the conjecture that every graph with exactly one Hamilton cycle must have a vertex of degree less than 4. Herbert informed us that he has found a class of counterexamples to the latter conjecture. The smallest one available is a 2-connected graph with 338 vertices (all of which, except a dozen, have degree 4). It is possible to construct a 3-connected counterexample with 478 vertices. The 4-connected case is the subject of Conjecture 1.

Locally connected graphs

Ryjáček has conjectured that every locally connected graph is weakly pancyclic (has cycles of all lengths between 3 and the circumference). A proof is only available for planar triangulations (= locally 2-connected planar graphs). This suggests the following two weaker conjectures.

Conjecture 2 *Every locally 2-connected graph is weakly pancyclic.*

Conjecture 3 *Every locally connected planar graph is weakly pancyclic.*

Stable dominating cycles

A cycle C in a graph is *stable* if no other cycle covers $V(C)$. Kochol has constructed examples of cubic graphs with stable dominating cycles. All of these graphs contain Petersen minors.

Question 4 (Kochol) *Is there a Petersen-minor-free cubic graph with a stable dominating cycle?*

Restricted dominating cycles

As proved by Fleischner and Kochol, and independently Kužel, if the dominating cycle conjecture is true, then one may even prescribe two edges of a dominating cycle in any essentially 4-edge-connected cubic graph.

Conjecture 5 (Kaiser) *Every essentially 4-edge-connected cubic graph has a dominating cycle avoiding any given vertex.*

Conjecture 6 (Hao Li) *Every essentially 4-edge-connected cubic graph has a dominating cycle through any 3 given independent edges.*

Conjecture 7 *Every essentially 4-edge connected cubic graph has a dominating cycle through any path of 3 edges.*

Question 8 (Kochol) *If G is essentially 4-edge-connected and cubic, can one find a Hamilton cycle through any 2 edges in $L(G)$ (assuming the Thomassen conjecture holds true)?*

More on dominating trails

Question 9 (Hao Li) *Does there exist, in every essentially 6-edge-connected graph, a closed trail that dominates all vertices of degree ≥ 4?*

Question 10 *Does every 6-connected line graph have a cycle spanning all vertices of degree ≥ 7? Equivalently: does every essentially 6-connected graph possess a closed trail dominating all edges of weight ≥ 7?*

Question 11 (Kriesell) *Is it true that every essentially 4-connected graph G, each of whose vertices of degree 3 is contained in a cycle of length at most 4 [or: at most 5], has a dominating closed trail?*

One-tough graphs

Conjecture 12 (Kriesell) *If G is 1-tough, then $G[K_2]$ (the lexicographic product with K_2) is pancyclic.*

(The hamiltonicity of $G[K_2]$ is equivalent to a conjecture of Jackson and Wormald (1990) that every 1-tough graph has a spanning walk visiting every vertex at most twice. So even this is open.)

A 1-tough graph is *minimally 1-tough* if for every edge e, $G-e$ is not 1-tough.

Conjecture 13 (Kriesell) *Every minimally 1-tough graph has a vertex of degree 2.*

Bi-claw-free graphs

The *bi-claw* is the unique tree consisting of six leaves and two degree 4 vertices. A graph is *bi-claw-free* if it contains no induced bi-claw. Hao Li has conjectured that there is a constant C such that every balanced bipartite connected bi-claw-free graph G with minimum degree $\geq C$ is hamiltonian.

Question 14 (Kriesell) *Is there an analogue of the Ryjáček closure for bi-claw-free bipartite graphs?*

The *I-tree* is the tree with 4 leaves and 2 vertices of degree 3. The following is a weakening of the above conjecture of Hao Li.

Conjecture 15 *There is a constant C such that every balanced bipartite connected graph without any induced I-tree and with minimum degree $\geq C$ is hamiltonian.*

Question 16 (Hao Li) *Does every connected balanced bipartite graph G with $\delta(G) \geq 3$ and without induced I-trees have a dominating circuit?*

(The existence of a dominating closed *trail* follows from a theorem of H. J. Lai [Graphs whose edges are in small cycles, *Discrete Math.* **94** (1991), 11–22].)