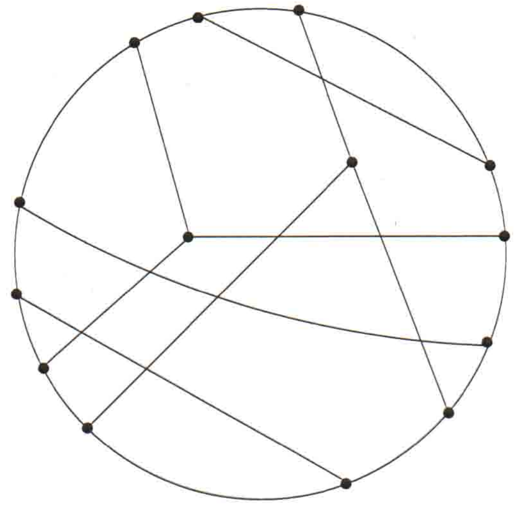
A COUNTEREXAMPLE TO THE BIPARTIZING MATCHING CONJECTURE

NZ5FC: Every bridgeless graph admits a nowhere zero 5-flow.

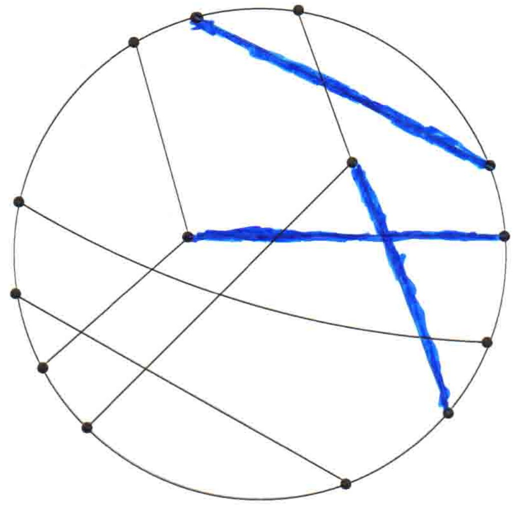
CDCC: Every bridgeless graph G contains a collection S of cycles such that each edge of G is covered by exactely two cycles of S.

A dominating cycle D in a cubic graph G is a cycle such that for every edge $xy \in E(G)$, $\{x,y\} \cap V(D) \neq \emptyset$.

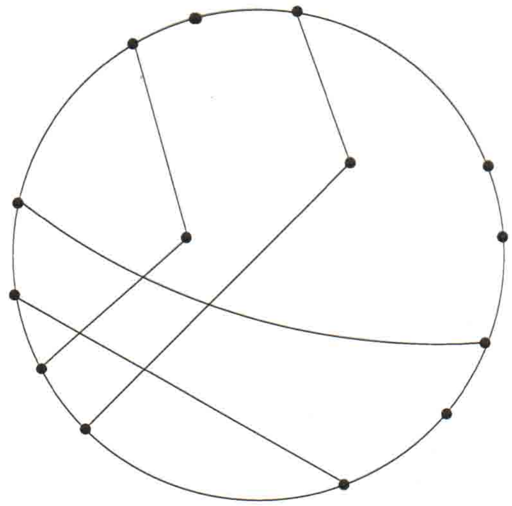
Let (G, D) be a cubic graph with a dominating cycle D, then a matching M in $E(G)-E(D)$ which covers $V(G)-V(D)$, is called a BM if $G-M$ is a cycle or a subdivision of a bipartite cubic graph.



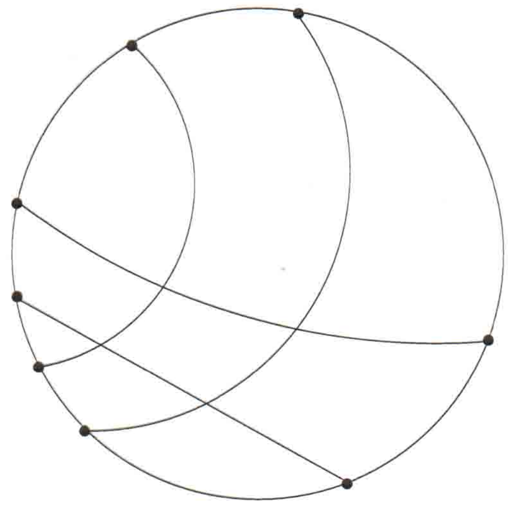
(G, D)



(G, D) with a BM



(G, D) − BM



Bipartite graph

Theorem (Fleischner, Stiebitz): Every (G,D) has a BM.

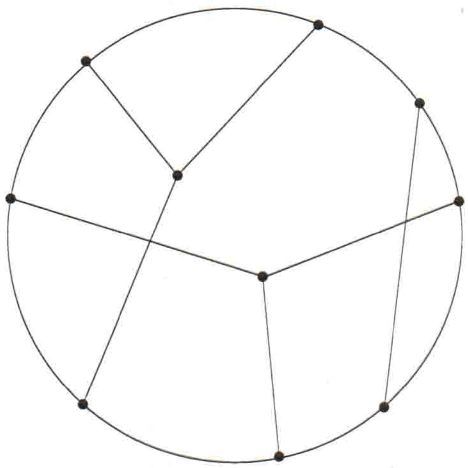
Another proof was given by: Dvorak, Kral, kaiser

Theorem (Fleischner): Let (G,D) have 2 disjoint BMs, then G has a NZ5F and a CDC.

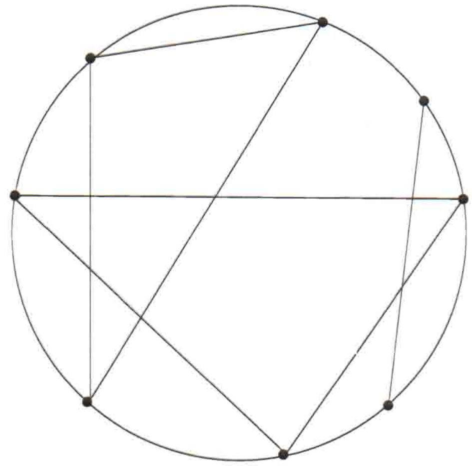
Bipartizing Matching Conjecture (BMC) (Fleischner): Let (G,D) be a snark, then (G,D) has 2 disjoint BMs w.r.to D.

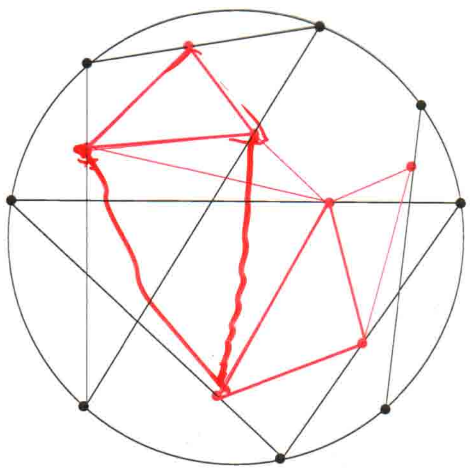
Modified BMC (Fleischner): Let (G,D) be a snark, then (G,D) has a dominating cycle D', such that (G,D') has 2 disjoint BMs.

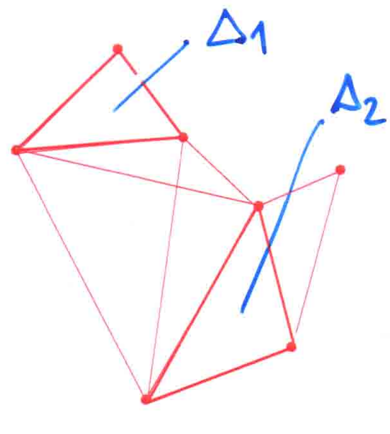
THE "CIRCLE-GRAPH" OF (G,D)



(G,D)







$(G,D)^c$

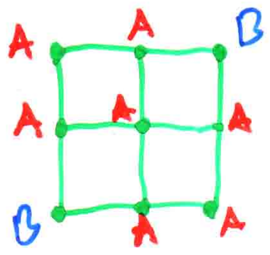
Every BM in (G,D) corresponds to an induced eulerian subgraph A in $(G,D)^c$ such that $|A \cap \Delta_i| = 1 \ \forall i$.

Theorem (Hoffmann): (G, D) has 2 disjoint BMs

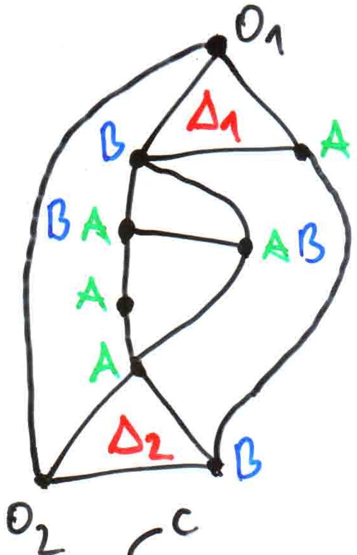
$$\Longleftrightarrow$$

From every Δ_i in G^c, one vertex o_i can be removed, such that all vertices of $G^c - \cup o_i$ can be covered by $A \cup B$, where A and B are induced eulerian subgraphs in $G^c - \cup o_i$, and $|V(A) \cap V(\Delta_i)| = 1$ and $|V(B) \cap V(\Delta_i)| = 1$.

Theorem (Gallai): Every graph H contains induced eulerian subgraphs A, B such that $V(A) \cup V(B) = V(H)$ and $A \cap B = \emptyset$.



H



$\underline{G^c}$

An **anti-eulerian** graph is a graph where the degree of every vertex is odd.

Question: Has every graph H two induced anti-eulerian subgraphs A, B such that $V(A) \cup V(B) = V(H)$

$\overline{C_{2n+1}}$

Graphs without covering

Theorem: (Hoffmann) A graph H has two induced anti-eulerian subgraphs A, B such that $V(A) \cup V(B) = V(H)$

$(\Leftrightarrow)$

$\overline{H}$ has two induced eulerian subgraphs A', B' such that $V(A') \cup V(B') = V(\overline{H})$ and $|A|$ and $|B|$ are even.

$i = 1, 2$. By Proposition 2.9 this is equivalent to showing that $(G_{M_i}, D_{M_i})^c$ is Eulerian, $i = 1, 2$. By the definition of g and the definition of the sets A, B, O (see Def. 2.11) it follows, see Figure 2, that

$$(G_{M_1}, D_{M_1}) = ((G^* - B^*)_{O^*}, (D - B^*)_{O^*}) .$$

Since

$$(G_{M_1}, D_{M_1})^c = ((G^* - B^*)_{O^*}, (D - B^*)_{O^*})^c = \langle A \rangle_{G^c}$$

and $\langle A \rangle_{G^c}$ is Eulerian by definition of a $\triangle$-colouring and because $C = \emptyset$, $g^{-1}(A)$ is a BM in G. The same arguments hold for B. This finishes the proof of the theorem.

3 Construction of a counterexample

By Theorem 2.12 a counterexample to the BMC has no $\triangle$-colouring. Let (G, D) have chords. If we want to know whether G^c has a $\triangle$-colouring w.r. to T, we can proceed in the following manner:

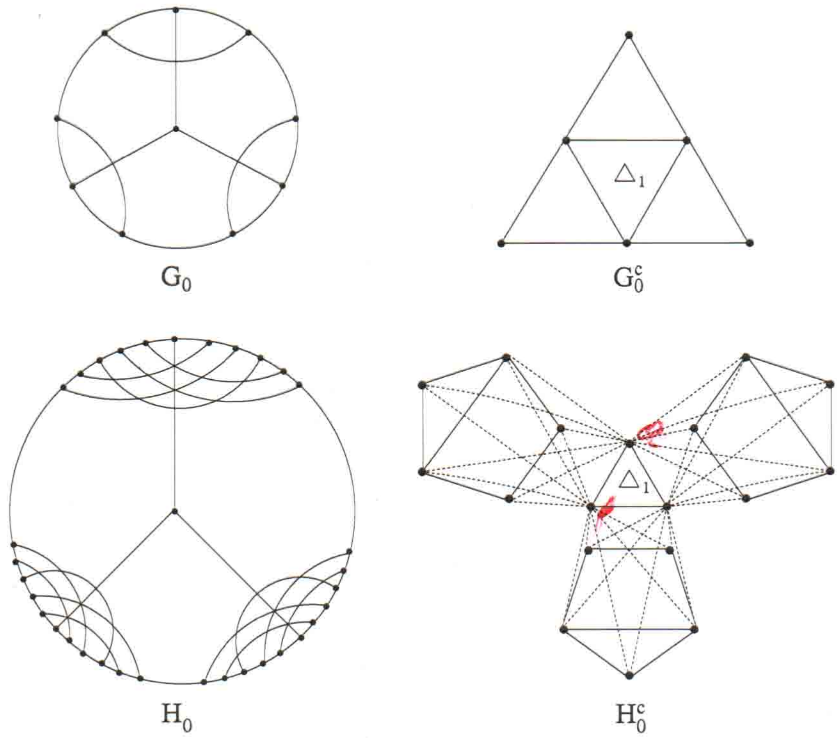
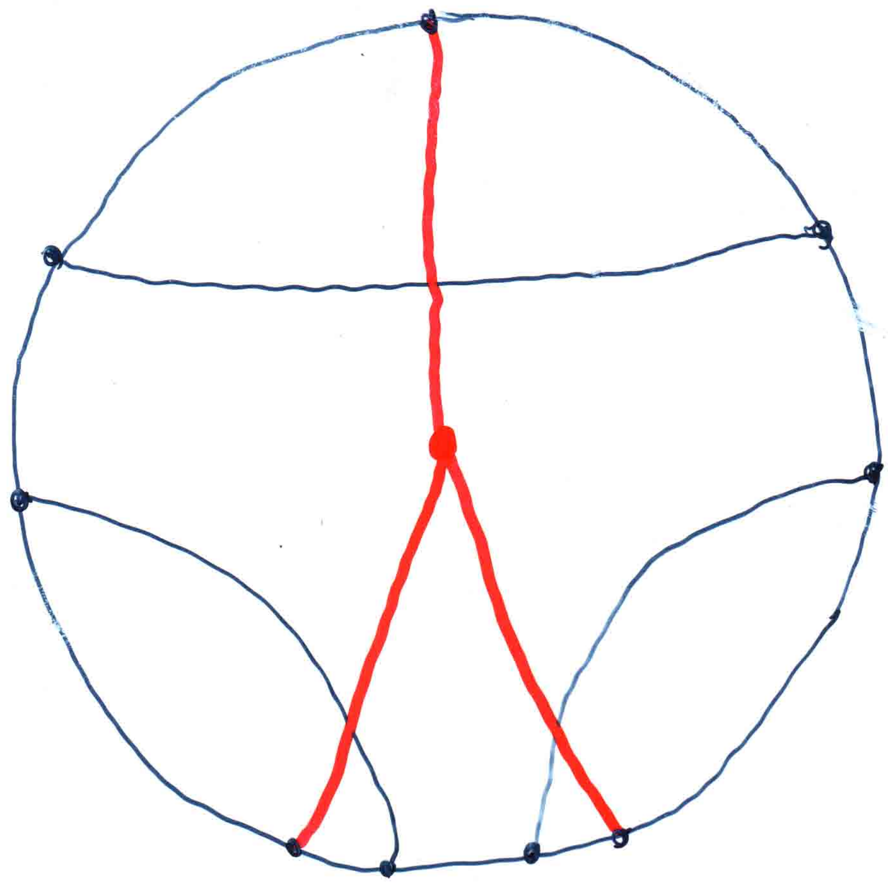
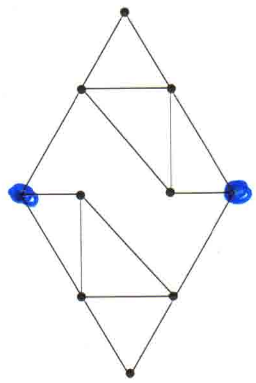
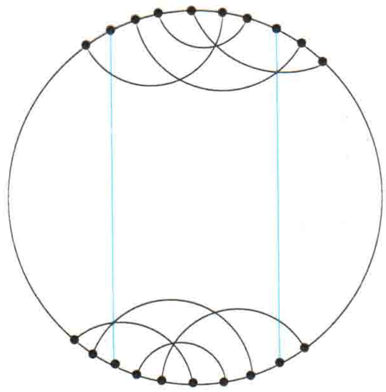
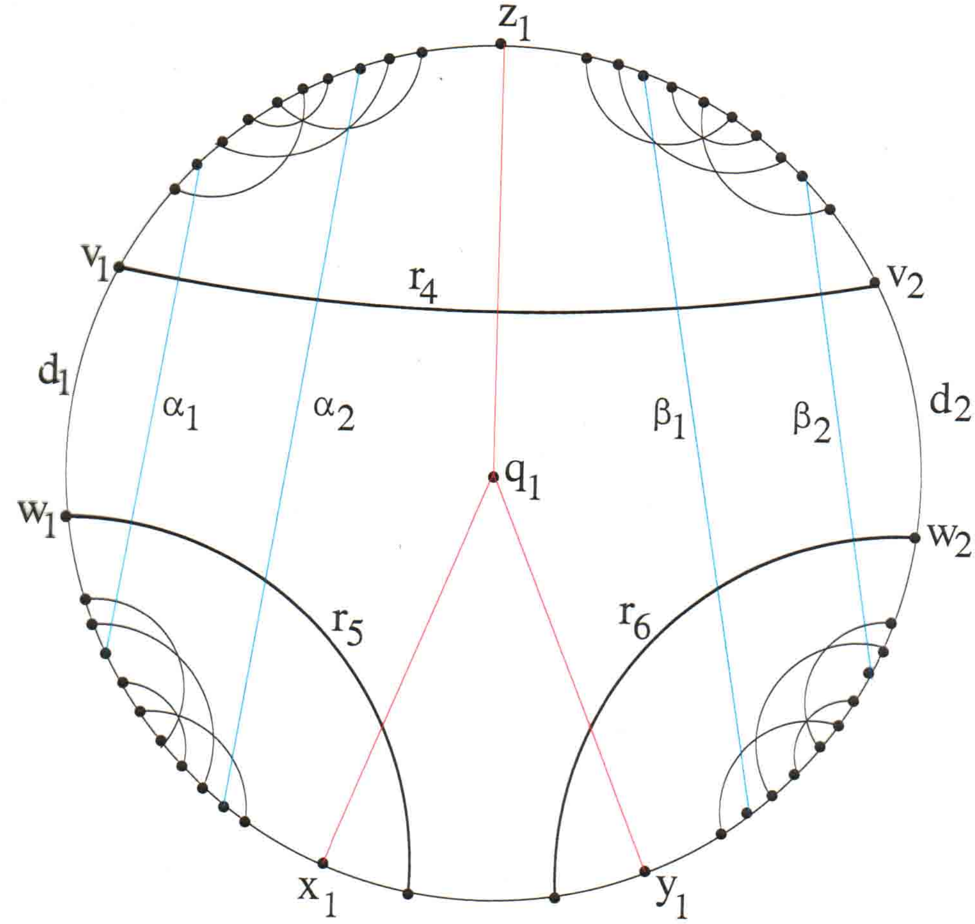


Figure 4: Two cyclically 3-edge-connected graphs without 2dBMs.









z_1
v_1
r_4
v_2
d_1
α_1
α_2
β_1
β_2
d_2
q_1
w_1
w_2
r_5
r_6
x_1
y_1

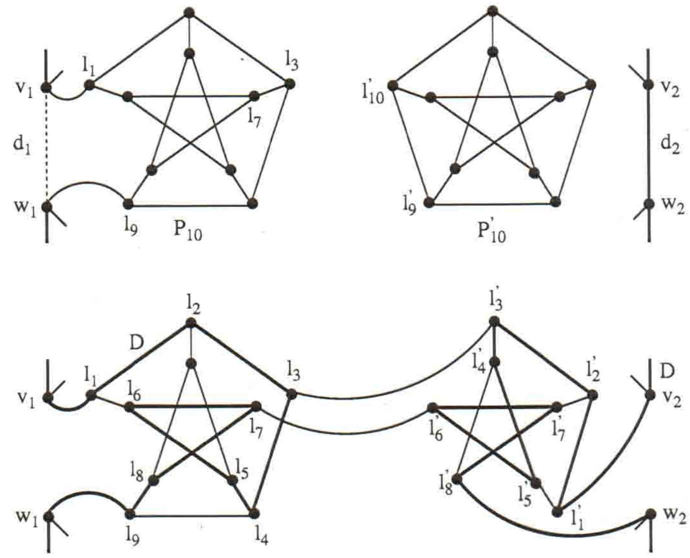
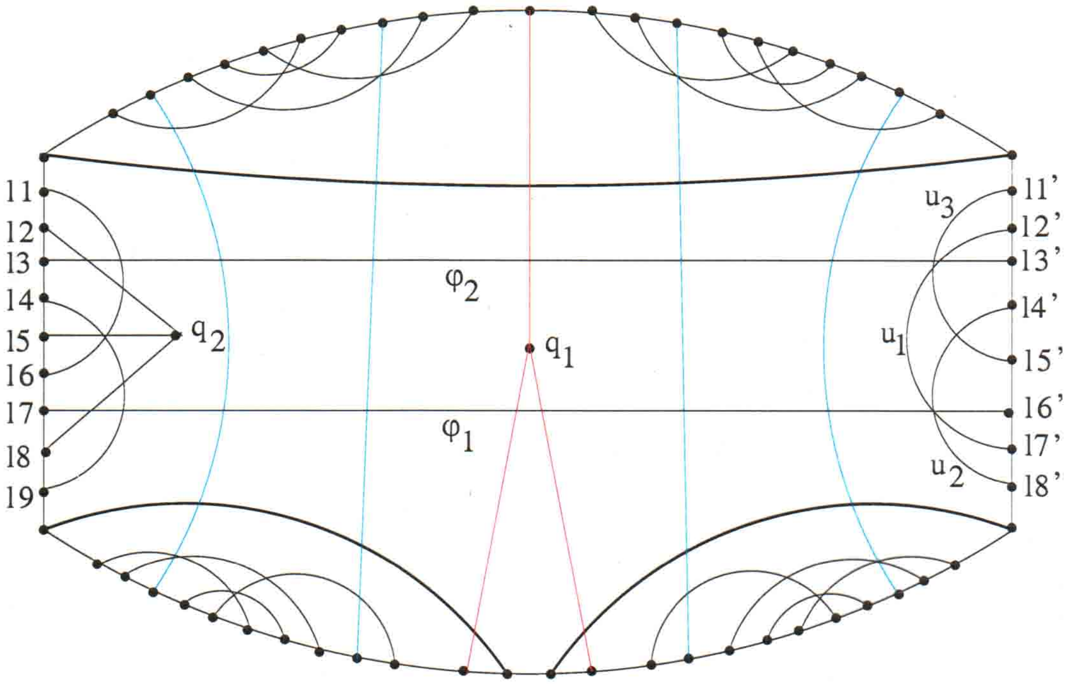
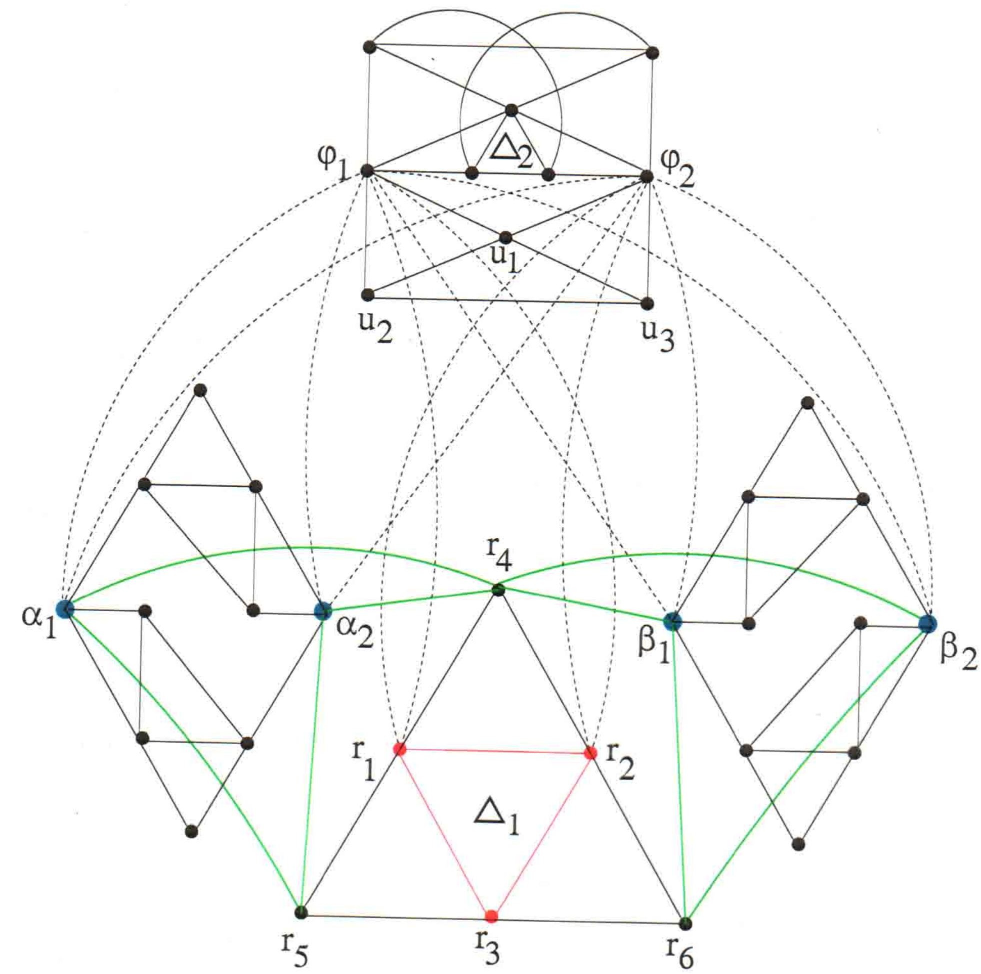


Fig. 9. The transformation from the graph J into the graph Y.



l1
l2
l3
l4
l5
l6
l7
l8
l9
q2
φ2
q1
φ1
l1'
l2'
l3'
l4'
l5'
l6'
l7'
l8'
u3
u1
u2



Δ_2
φ_1
φ_2
u_1
u_2
u_3
r_4
α_1
α_2
β_1
β_2
r_1
r_2
Δ_1
r_5
r_3
r_6

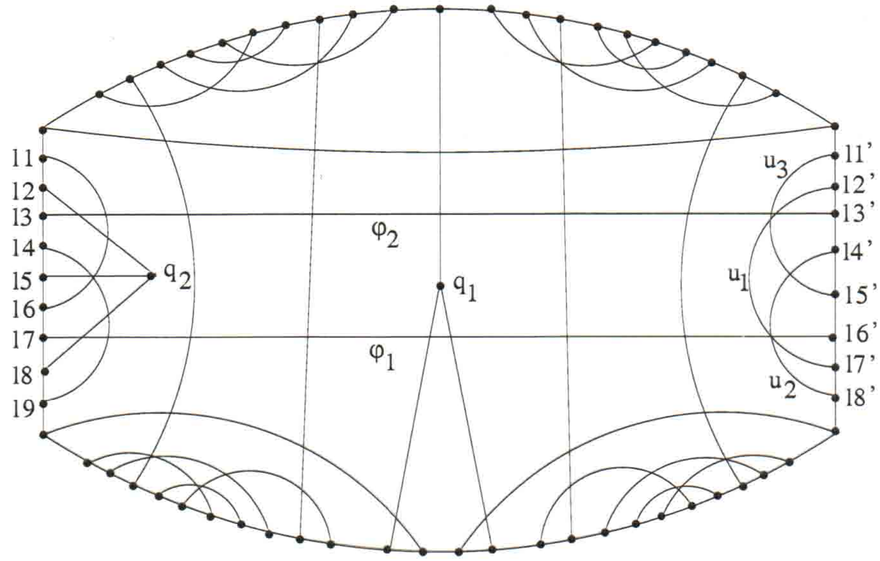


Figure 9: The graph (Y, D) without 2dBMs.

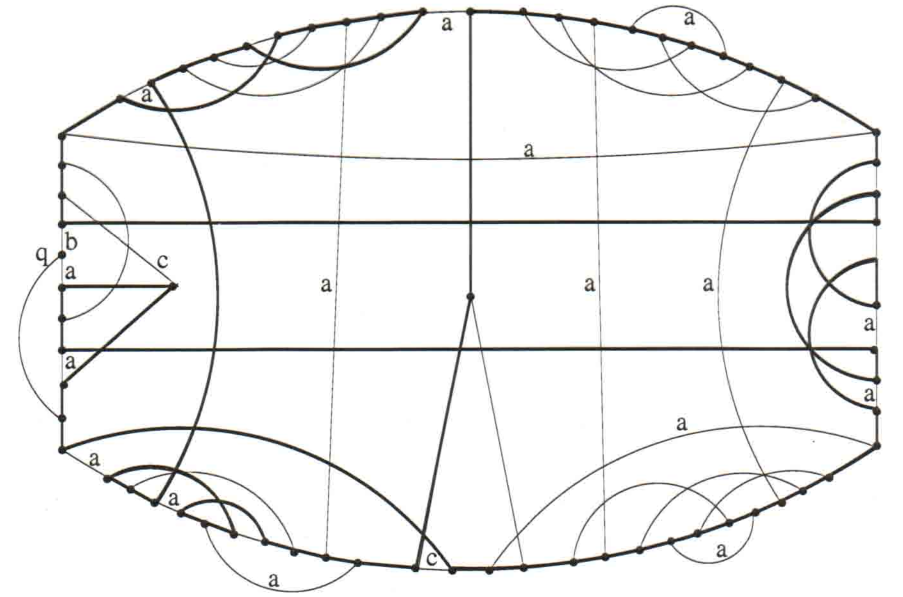
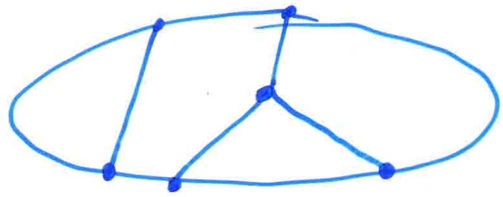


Figure 10: The graph (Y, D_1) with 2dBMs. The edges in bold face show D_1.

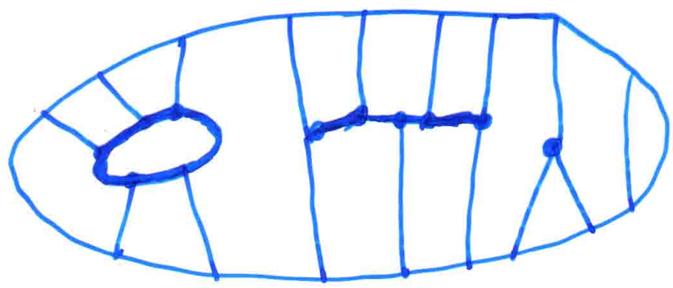
Comments on the DCC

Let G be a cubic graph with $\delta_c \geq 4$.

DCC: G has a dominating cycle.



weaker DCC: G has a matching M such that G/M is hamiltonian.



Question: Is there a number k such that every G has a cycle C such that the distance between every vertex of G and C is smaller than k?