2-connected claw-free graphs are prism-hamiltonian

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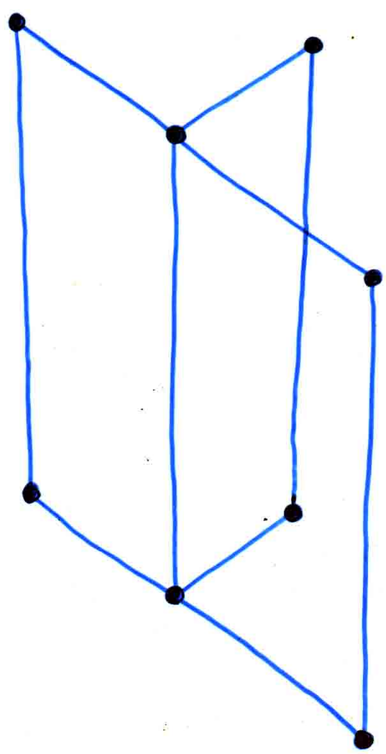
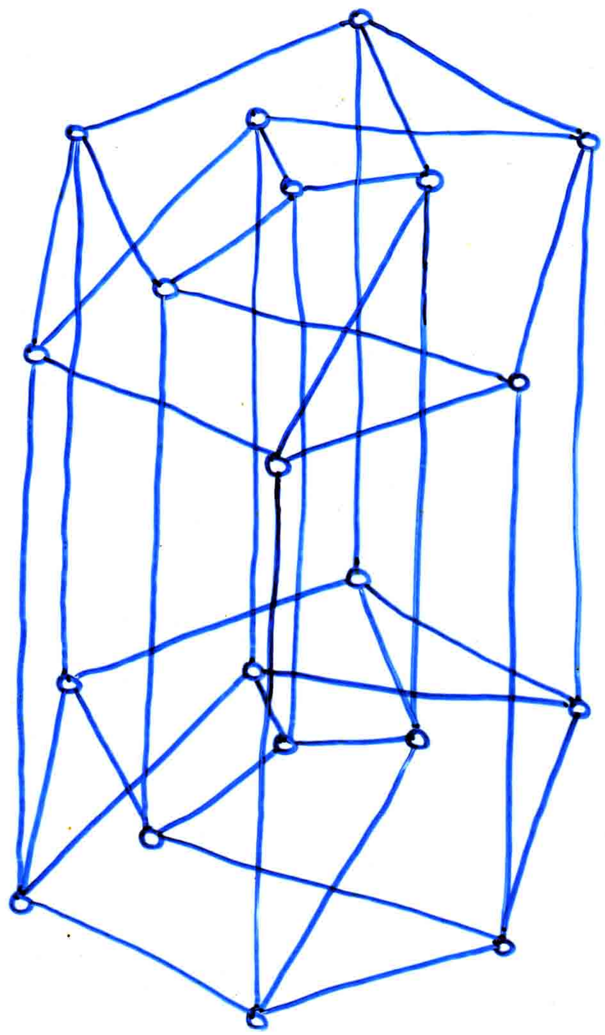


prism over $G = G \square K_2$



Conjecture

Any 2-connected claw-free graph is prism-hamiltonian.

[in:
Hamilton cycles in prisons over graphs
by Kaiser, Král', Rosenfeld, Ryjáček, Voss]

k-tree – spanning tree with all vertices of degree at most $\underline{k}$

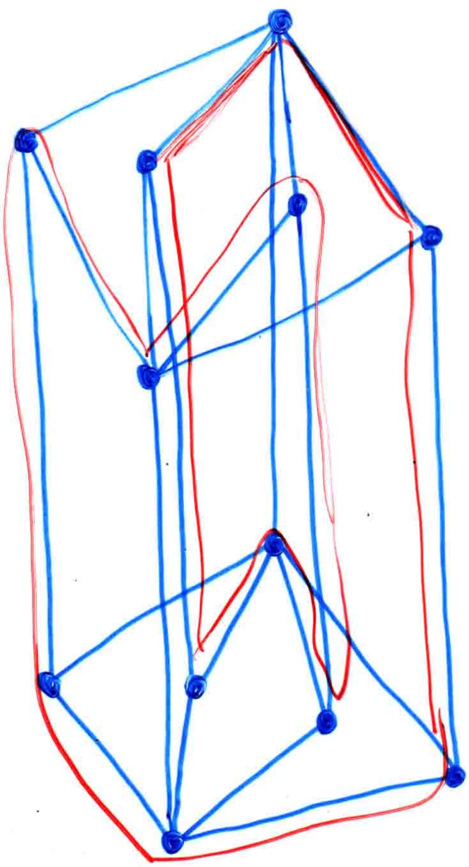
2-tree = Hamilton path

k-walk – spanning closed walk visiting no vertex more than $\underline{k}$ times

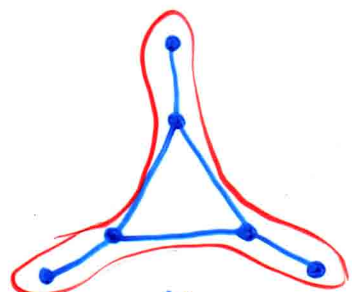
1-walk = Hamilton cycle

2-tree $\Rightarrow$ hamiltonian prism $\Rightarrow$ 2-walk

$\not\Leftarrow$ $\qquad\qquad$ $\not\Leftarrow$



$K_{2,4} \square K_2$



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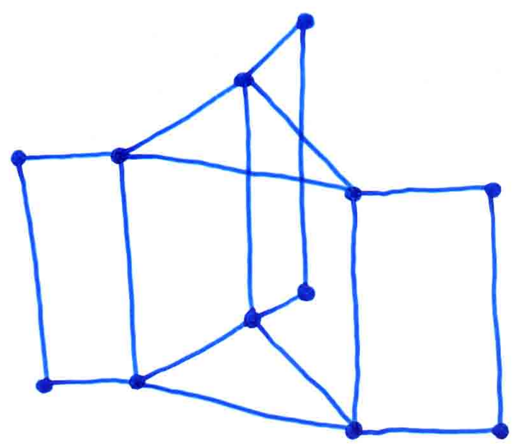
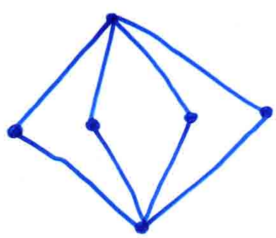




Thm (Paulraja)

$G \square K_2$ is hamiltonian $\iff$

G has an SEEP-subgraph

Proof

Lemma The prism over any even cactus C with $\Delta(C) \leq 3$ is hamiltonian.

P induction on the number of vertices of C

- $C \square K_2$ has a Hamilton cycle F s.t. F contains the edge $\underline{xx^*}$ for each degree 2 vertex $\underline{x}$ belonging to a leaf of C

- $|V(C)| > 2$, C is not a cycle
- contract each cycle of C to a vertex $\Rightarrow$ tree T
- $\underline{t}$ of degree 1 in T
  - t not on a leaf

- $t = v_Q$

A subgraph $\underline{H}$ of a connected graph $\underline{G}$ is called an EP-subgraph of G if

1. $\underline{H}$ is a connected spanning subgraph of $\underline{G}$,
2. $\Delta(H) \leq 4$, and
3. $H = E \cup P$, where
   - E is an edge-disjoint union of cycles,
   - P is a vertex-disjoint union of paths, s.t. no vertex of a path of P is of degree 4 in H, and
   - E and P are edge-disjoint

EEP-subgraph

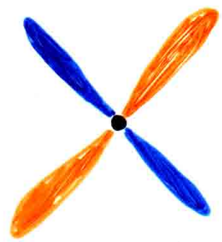
- H is an EP-subgraph of G
- duplicate the edges of the paths of P
- => eulerian multigraph H' of maximum degree 4
- ¿ even cycle decomposition ?

  YES - EEP-subgraph

consider an EEP-subgraph H

it admits an even cycle decomposition

=> it is possible to bicolor the edges of each of the even cycle



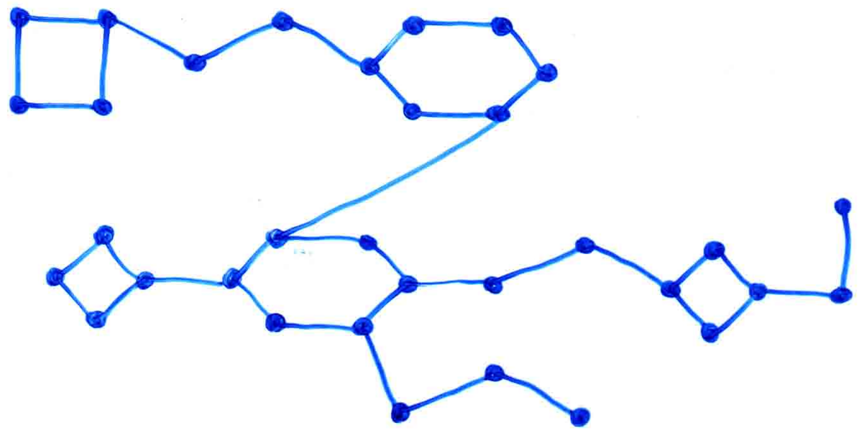
traversal rules:

- Any vertex can be used as the starting vertex of the tour
- If an edge of one color is used to reach a vertex and there is another edge of the same color incident with the vertex, then another edge of the same color must be used to leave the vertex

↳ AR-tour (eulerian tour)

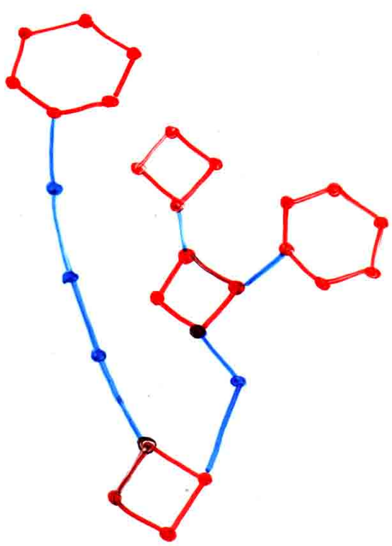
↳ SEEP-subgraph

cactus

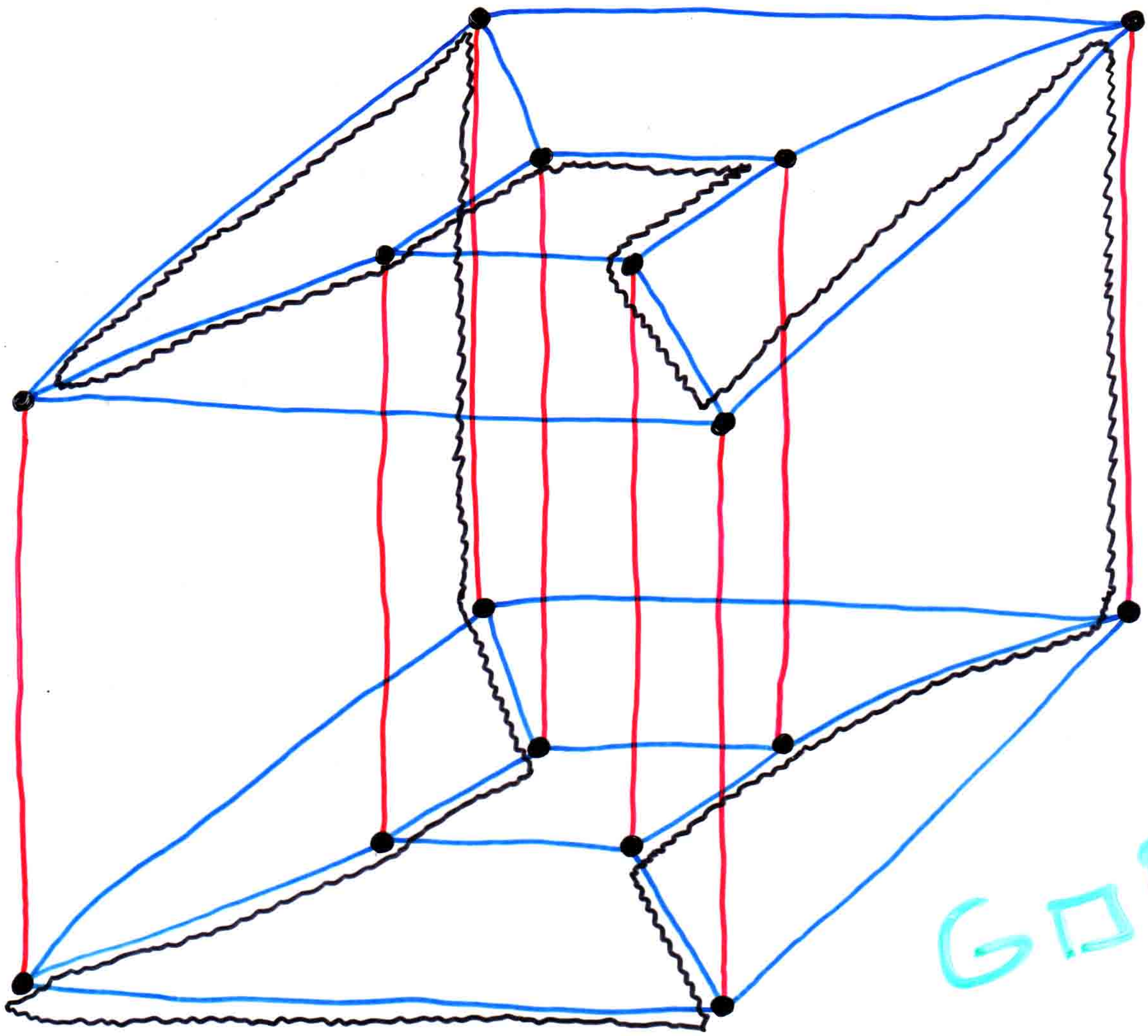


- two cycles are vertex-disjoint
- every vertex of degree at least 3 lies on a cycle
- has at least 2 vertices

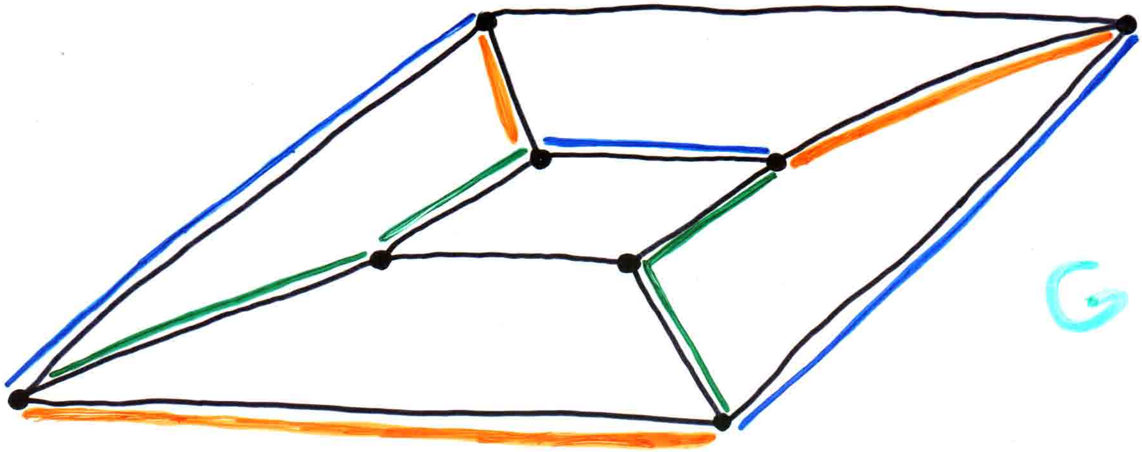
leaf



even cactus



$G \square P_2$



G

Theorem (Kaiser, Kráľ, Rosenfeld, Ryjáček, Voss)

Every 2-connected line graph is prism-hamiltonian.

Proof

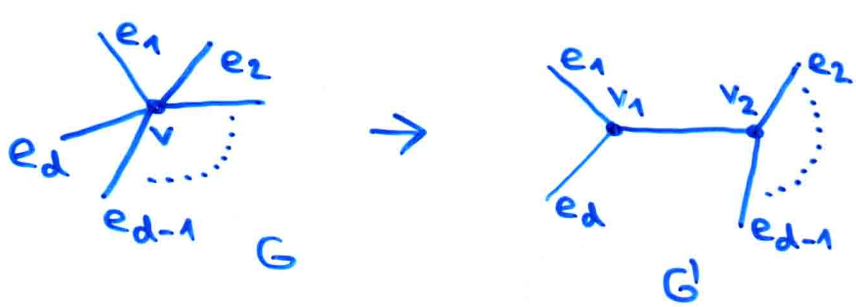
- contraction of a subtree $T \subset G$ - preserving multiple edges

Lemma G bridgeless multigraph with minimum degree $\delta \geq 3$

$\Rightarrow$ $\exists$ a cubic bridgeless multigraph G_3 s.t. G can be obtained by the contraction of some pairwise disjoint induced subtrees of G_3

P

- excess $exc(G) = \sum_{v \in V(G)} (\deg_G(v) - 3)$
- induction on $exc(G)$
- $exc(G) = 0 \Rightarrow G$ is cubic
- $exc(G) > 0$: vertex $\underline{v}$, $\deg_G(v) > 3$



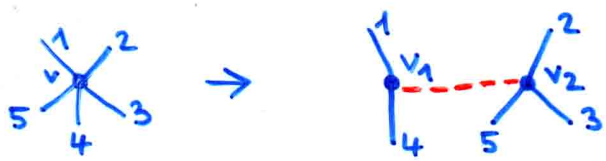
$exc(G') = exc(G) - 1$

- apply induction to G'

Thm Let G be a multigraph.
If $L(G)$ is 2-connected, then the prism over $L(G)$ contains a Hamilton cycle.

P
- G is a star OK
- G is not a star
  - remove pendant edges $\to G^-$ ... bridgeless
  - all the vertices of G^- have even degrees $\Rightarrow$ Euler tour $\Rightarrow L(G^-)$ hamiltonian (and $L(G)$ also)
  - $\exists$ vertex of odd degree
  - suppress all degree 2 vertices $\to G^=$ ... bridgeless, $\delta(G^=) \geq 3$
  - $\exists$ eulerian factor F in $G^=$, k ... # of its components
  - color edges of F black

splitting of a vertex v



detachment of an edge uv from v

- let F be 2-regular
- $k-1$ edges joining F to the connected subgraph ... red
- $\exists$ edge o neither red nor black ... orange - detach o
- the other edges ... brown - detach brown edges

$\Rightarrow$ coloring of G

Lemma G bridgeless multigraph with $\delta \geq 3$

$\Rightarrow$ G contains an eulerian factor G' s.t. the degree of each vertex is non-zero in G'

P

eulerian factor = a spanning (not necessarily connected) subgraph of G with all degrees even

- construct G_3
- Petersen theorem $\Rightarrow$ 1-factor
- complement of a 1-factor is a 2-factor in G_3
- E ... set of the edges of the 2-factor $\cap\, E(G)$
- subtree $T_v \Leftrightarrow$ vertex v
- # edges of the 2-factor incident with T_v (v) is even, $\neq 0$

$\Rightarrow$ E is the desired eulerian factor of G

CLAW-FREE CLOSURE (Z. Ryjáček)

G claw-free

locally connected vertex

local completion operation

closure of G : cl(G)

- cl(G) uniquely determined
- cl(G) is a line graph of a triangle-free graph
- G is hamiltonian <=> cl(G) is hamiltonian
- (S. Brandt, O. Favaron, Z. Ryjáček)
  G is traceable <=> cl(G) traceable
- (Z. Ryjáček, A. Saito, R.H. Schelp)
  - cl(G) has a 2-factor with $\underline{k}$ components =>
    G has a 2-factor with at most $\underline{k}$ components
  - G is covered by $\underline{k}$ cycles <=> cl(G) is covered by $\underline{k}$ cycles
- (S. Ishizuka)
  - G has a path-factor with $\underline{k}$ components <=>
    cl(G) has a path-factor with $\underline{k}$ components
  - G is covered by $\underline{k}$ paths <=> cl(G) is covered by $\underline{k}$ paths

Lemma (Broersma, Trommel)

Let G be a graph and let $\{x, y, u, v\}$ be a subset of four vertices of V s.t.

- $uv \notin E(G)$
- $\{x, y\} \subseteq N(u) \cap N(v)$.

If

- $N(x) \subseteq N[u] \cup N[v]$,
- $N(y) \setminus N[x]$ induces a complete graph (or is empty),

then

for every cycle C' in $G+uv$ there exists a cycle C in G s.t. $V(C') \subseteq V(C)$.

→ K_4^*-closure of G

- G is hamiltonian $\Leftrightarrow$ K_4^*-closure of G is hamiltonian
- for a claw-free graph G

$$cl_R(G) \subseteq K_4^*(G)$$

- $\forall$ claw-free graph G $\exists$ K_4^*-closure of G s.t.

$$cl_R(G) = K_4^*(G)$$

Lemma Let G be a graph.

Let $\{x, y, u, v\}$ satisfy

- $uv \notin E(G)$, $\{x, y\} \subseteq N(u) \cap N(v)$,
- x is a claw-free vertex
- $N(y) \setminus N[x]$ induces a complete graph (possibly empty)

Then G admits a LSEEP-subgraph

$\iff$ $G + uv$ does.

LSEEP = SEEP s.t. every pair of cycles share at most one vertex